

DERIVED \mathbb{C} -ANALYTIC GEOMETRY I: GAGA THEOREMS

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ABSTRACT. We further develop the foundations of derived \mathbb{C} -analytic geometry introduced in [11] by J. Lurie. We introduce the notion of coherent sheaf on a derived \mathbb{C} -analytic space. Moreover, building on the previous joint work with T. Y. Yu [20], we introduce the notion of proper morphism of derived \mathbb{C} -analytic spaces. We show that these definitions are solid by proving a derived version of Grauert's proper direct image theorem and of the GAGA theorems. The proofs rely on two main ingredients: on one side, we prove a comparison result between the ∞ -category of higher Deligne-Mumford analytic stacks introduced in [20] and the ∞ -category of derived \mathbb{C} -analytic spaces. On the other side, we carry out a careful analysis of the analytification functor introduced in [11] and prove that the canonical map $X^{\text{an}} \rightarrow X$ is flat in the derived sense. This is part of a series of articles [18, 19] that will soon appear.

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INTRODUCTION

Since its appearance, derived algebraic geometry has been proved an useful tool to deal with situations where the techniques of classical algebraic geometry couldn't yield a full understanding of the involved mathematical phenomena. Striking examples are given by the work of B. Toën on derived Azumaya algebras [23] and by the approach to the geometric Langlands program of D. Gaitsgory [7]. Other applications can be found in Gromov-Witten theory, see for example the papers [21, 16]. On the other side, derived algebraic geometry has been an active research field on its own, as the recent works on shifted symplectic and Poisson structures [17, 3] demonstrate.

However, before reaching the foundations nowadays accepted [26, 12, 13, 14], there has been several different approaches and attempts, see e.g. [4]. There has been recent activity toward the possibility of having a derived version of \mathbb{C} -analytic geometry, but the state of the art still resembles very much the one of derived algebraic geometry a decade ago. In [2], the authors propose an approach to analytic geometry from the point of view of relative algebraic geometry [25]. As far as we understand, this would amount to take the affine objects to be simplicial ind-Banach rings. This is indeed a reasonable proposal, which is, however, quite different from the perspective we are adopting in the present work.

Pregeometries and derived geometry. The approach to derived \mathbb{C} -analytic geometry used in this article is due to J. Lurie and relies on his general theory of pregeometries developed in [12]. He proposed to use it to lay foundations of derived \mathbb{C} -analytic geometry in [11, §11, §12]. As this way of constructing a category of derived objects differs significantly from the perhaps more familiar one of [26], it seems worth to describe the general ideas.

The procedure of [26] consists in taking as input an ambient “linear” ∞ -category \mathcal{C} endowed with a symmetric monoidal structure and satisfying certain basic assumptions. Out of this ambient ∞ -category \mathcal{C} one obtains a category of affine objects Aff , which precisely coincides with the ∞ -category of commutative monoids in \mathcal{C} (with respect to the tensor structure). Then, one endows this category of affine objects with some geometrical extra structure, such as a Grothendieck topology τ and a collection of “smooth” morphisms \mathbf{P} . These data are required to satisfy a certain amount of compatibility conditions (we refer to [26, § 1.3.2] for the precise formulation of these ideas). At this point, one can form the ∞ -category of hypercomplete sheaves $\text{Sh}(\text{Aff}, \tau)^\wedge$, and inside this category it is possible to isolate objects that behave much like schemes, algebraic spaces, Deligne-Mumford stacks etc. known as *geometric stacks*. All of this is applied to derived algebraic geometry by taking \mathcal{C} to be the ∞ -category sMod_A of simplicial A -modules, where A is any (discrete) commutative ring. Commutative monoids in sMod_A can be identified with simplicial commutative A -algebras, whose underlying ∞ -category precisely coincides with the (opposite of the) category of affine derived schemes.

However, in developing derived analytic geometry it is not so clear what the ambient ∞ -category \mathcal{C} should be. Possible attempts are the category of simplicial ind-Banach spaces, or of simplicial bornological spaces (we refer to [1] where it is shown that it is indeed possible to endow such categories with the structure of HAG contexts). Still, the framework of [26] can be successfully applied as soon as the category of affine objects is known. Together with T. Y. Yu, in [20] we showed that the Grauert proper direct image theorem and the GAGA theorems hold for (underived) higher Artin analytic stacks (see also [24] for a different kind of applications in a very similar framework). But, again, it is not at all clear what should the affine objects of derived \mathbb{C} -analytic geometry look like, though some version of analytic rings [6], could yield a workable setting.

The approach taken by J. Lurie in [12] is quite different and it can be used to deal with derived \mathbb{C} -analytic geometry, and we hope that this paper will contribute to show the solidity of the foundations laid in [11]. The essential idea of [12] is that we do not need an ambient linear category to develop derived algebraic geometry, nor we need to start with a well-known (∞ -)category of affine objects: we only need

a category \mathcal{C} of classical geometrical objects that we declare to be smooth, together with a collection of smooth morphisms between such objects. With only this, it is possible to construct the “free category of derived \mathcal{C} -objects” simply by requiring that pullbacks along smooth morphisms have to be classical (i.e. computed in \mathcal{C}). The correct way of formalizing this striking idea is given with the language of pregeometries. We refer the reader to Section 1.1 (and obviously to [12]) for a more detailed technical account on this notion. For the time being, let us say that a pregeometry \mathcal{T} is an ∞ -category with finite products, equipped with a Grothendieck topology τ and a collection of admissible morphisms (these data are required to satisfy some mutual compatibility conditions, see Definition 1.1). The reader could think of this as a (multisorted) Lawvere theory equipped with some geometrical extra structure, and indeed the case of multisorted Lawvere theories is covered by the theory of [12] when the Grothendieck topology is discrete and the admissible morphisms are precisely the equivalences. As in the case of a (multisorted) Lawvere theory, there is a notion of a “model” for \mathcal{T} . This concept is referred to in [12] as \mathcal{T} -structures. More specifically, if \mathcal{X} is an ∞ -topos (the reader might want to think to the ∞ -category of spaces \mathcal{S} or to sheaves of spaces on some topological space), a \mathcal{T} -structure on \mathcal{X} is a product preserving functor $F: \mathcal{T} \rightarrow \mathcal{X}$ satisfying the following additional conditions:

- (1) the functor F preserves the pullbacks in \mathcal{C} of the form

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ X' & \longrightarrow & X \end{array}$$

where f is an admissible morphism.

- (2) Whenever a family of morphisms $\{U_i \rightarrow X\}$ generates a covering sieve, the induced morphism

$$\coprod F(U_i) \rightarrow F(X)$$

is an (effective) epimorphism in the ∞ -topos \mathcal{X} .

The full subcategory of $\text{Fun}(\mathcal{T}, \mathcal{X})$ spanned by \mathcal{T} -structures will be denoted $\text{Str}_{\mathcal{T}}(\mathcal{X})$. However, as in the case of locally ringed spaces, we do not really wish to work with all the morphisms of \mathcal{T} -structures, but only with those that have a sufficiently regular behavior on the stalks of the topos \mathcal{X} . This can be achieved with the notion of local transformation: a morphism of \mathcal{T} -structures $\alpha: \mathcal{O} \rightarrow \mathcal{O}'$ is said to be *local*

if for every admissible morphism $f: U \rightarrow V$ in \mathcal{T} the induced square

$$\begin{array}{ccc} \mathcal{O}(U) & \xrightarrow{\mathcal{O}(f)} & \mathcal{O}(V) \\ \downarrow \alpha_U & & \downarrow \alpha_V \\ \mathcal{O}'(U) & \xrightarrow{\mathcal{O}'(f)} & \mathcal{O}'(V) \end{array}$$

is a pullback in \mathcal{X} . The subcategory of $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ spanned by local morphisms will be denoted by $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})$.

Before going any further, it is important to discuss a couple of examples, on which the intuition is based. The first important situation is the one of a *discrete* pregeometry (that is, of a multisorted Lawvere theory).

Example 1. Let k be a (discrete) commutative ring. We let $\mathcal{T}_{\mathrm{disc}}(k)$ be the opposite category of finitely presented free k -algebras. Admissible morphisms are precisely isomorphisms and the Grothendieck topology is the discrete one. In this case, $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{S}) = \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}^{\mathrm{loc}}(\mathcal{S})$ can be identified with the underlying ∞ -category of simplicial k -algebras. If $\mathrm{char}(k) = 0$ we can further interpret this as the category of \mathbb{E}_{∞} connective Hk -algebras, but it is important to remark that in positive (or mixed) characteristic we recover the simplicial formalism rather than the spectral one. Similarly, if \mathcal{X} is an ∞ -topos, $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}) = \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}^{\mathrm{loc}}(\mathcal{X})$ can be identified with the ∞ -category of sheaves of simplicial commutative algebras over \mathcal{X} .

Thus, interpreting in spaces gave us back the ∞ -category considered in [26] to start the construction of derived algebraic geometry. However, this procedure does not provide us with the geometrical extra structure needed in [26]. Let us try to modify the pregeometry $\mathcal{T}_{\mathrm{disc}}(k)$ in order to encode the Zariski or the étale topology:

Example 2. Let k be a (discrete) commutative ring. We let $\mathcal{T}_{\mathrm{Zar}}(k)$ (resp. $\mathcal{T}_{\mathrm{ét}}(k)$) be the category of standard Zariski open (resp. étale) maps to \mathbb{A}_k^n . We say that a morphism is admissible if it is an open Zariski immersion (resp. an étale map), and we consider the Zariski (resp. étale) topology. In this case, the forgetful functors

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{Zar}}(k)}(\mathcal{S}) \rightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}}(\mathcal{S}), \quad \mathrm{Str}_{\mathcal{T}_{\mathrm{ét}}(k)}(\mathcal{S}) \rightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}}(\mathcal{S})$$

are fully faithful and the essential image is the collection of simplicial k -algebras A such that $\pi_0(A)$ is local (resp. strictly henselian). Moreover,

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{ét}}(k)}^{\mathrm{loc}}(\mathcal{S}) \rightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{Zar}}(k)}^{\mathrm{loc}}(\mathcal{S})$$

is fully faithful, and a morphism $f: A \rightarrow B$ in $\mathrm{Str}_{\mathcal{T}_{\mathrm{Zar}}(k)}(\mathcal{S})$ lies in $\mathrm{Str}_{\mathcal{T}_{\mathrm{Zar}}(k)}^{\mathrm{loc}}(\mathcal{S})$ if and only if the induced $\pi_0(f): \pi_0(A) \rightarrow \pi_0(B)$ is a (local) morphism of local rings. The same analysis can be carried over a generic ∞ -topos.

Perhaps surprisingly, this example shows that when we interpret \mathcal{T}_{Zar} (or $\mathcal{T}_{\text{ét}}$) in spaces \mathcal{S} we do *not* recover the full ∞ -category of simplicial commutative rings. We only get back the ones with good locality behavior with respect to the Grothendieck topology we took into account. This suggests that the way of constructing the category of free \mathcal{T} -objects out of a pregeometry \mathcal{T} has to be slightly more complicated. As we said at the beginning, the category we are looking for should be determined by the requirement that pullbacks along admissible morphisms can be computed in \mathcal{T} , and it is freely generated by \mathcal{T} otherwise. This leads to the key of *geometry*, which is an ∞ -category \mathcal{G} having all finite limits and equipped with the very same geometrical data of a pregeometry (that is, a Grothendieck topology and a collection of admissible morphism). The notion of a structure for a geometry \mathcal{G} is modified accordingly: $\text{Str}_{\mathcal{G}}(\mathcal{X})$ is now the full subcategory of $\text{Fun}(\mathcal{G}, \mathcal{X})$ spanned by finite limit preserving functors which takes τ -covering to effective epimorphisms.

It is easy to imagine what a universal geometry generated by a pregeometry \mathcal{T} should be; namely, a continuous functor of Grothendieck sites $\varphi: \mathcal{T} \rightarrow \mathcal{G}$ which moreover preserves products and admissible pullbacks, and satisfies the following universal property: for every ∞ -topos \mathcal{X} , composition with φ should induce an equivalence

$$\text{Str}_{\mathcal{G}}(\mathcal{X}) \rightarrow \text{Str}_{\mathcal{T}}(\mathcal{X})$$

In this situation, we say that φ exhibits \mathcal{G} as a *geometric envelope* of \mathcal{T} . It can be shown that it always exists (see [12, Lemma 3.4.3]). It is considerably harder than Example 1 to show that when $\mathcal{T} = \mathcal{T}_{\text{Zar}}$, the geometric envelope is precisely the ∞ -category of finitely presented simplicial commutative rings, equipped with the (derived) Zariski topology. Nevertheless, it is true (see [12, Proposition 4.2.3]). Even more remarkably, it can be shown [12, Proposition 4.3.15] that the geometric envelope of $\mathcal{T}_{\text{ét}}$ is the ∞ -category of finitely presented simplicial commutative rings equipped with the (derived) étale topology.

So far we described only the procedure that allows to recover the affine objects, which, as we discussed at the beginning, is the only thing we need as the subsequent globalization step can be handled with the techniques of [26]. However, it is important to remark here that the framework of [12] comes with a gluing procedure that allows the author to introduce a structured space point of view on derived geometry. Roughly speaking, this consists in defining an ∞ -category of \mathcal{T} -structured topoi and then in isolating a full subcategory of \mathcal{T} -schemes inside. These categories are respectively denoted $\text{Top}(\mathcal{T})$ and $\text{Sch}(\mathcal{T})$, but we refer to the introduction of [18] for an expository account of these ideas. The reader who wish to see the details can consult [12, Definitions 1.4.8, 2.3.9]. In this work we will take the latter

point of view, but some comparison result with the approach stemming from [26] is provided (see Section 3).

The analytic pregeometry and the analytification functor. We are left to explain which pregeometry should lead to a meaningful notion of derived \mathbb{C} -analytic space. Following J. Lurie [11, 11.2, 11.3], we define \mathcal{T}_{an} to be the category of open subsets of \mathbb{C}^n , and declare a morphism to be admissible if it is an open immersion. Let us explain what \mathcal{T}_{an} -structures roughly look like. We hope this description will help the intuition of the reader in moving through this paper. For simplicity, we will think of \mathcal{T}_{an} -structures in Set rather than in the ∞ -category of spaces \mathcal{S} . First of all, it is important to observe that \mathcal{T}_{an} -structures are very similar to (local) \mathbb{C}^∞ -rings as they are introduced for example in [22]. A similar approach to \mathbb{C} -analytic geometry has been taken by [6], with the difference that they were taking a larger class of admissible pullbacks. Roughly speaking, the idea behind this Lawvere-style approach to \mathbb{C} -analytic geometry is that the affine objects ought to be related to (commutative) Banach algebras, but that the very topological nature of such objects prevents their category from having good categorical properties. However, we don't necessarily need the full structure of Banach algebra to give a working definition of \mathbb{C} -analytic spaces. What is really important is that Banach algebras admit a so-called holomorphic functional calculus, which is a formal way of encoding the action of the algebra of holomorphic functions on some open subset $U \subset \mathbb{C}^n$ on a given (commutative) Banach algebra A . Unraveling the definitions, it emerges that a \mathcal{T}_{an} -structure in Set is precisely the given of a commutative ring A together with the choice of a subset $A(U)$ of A^n for every open subset $U \subset \mathbb{C}^n$ and a choice of a map $A(U) \rightarrow A$ for every holomorphic function $U \rightarrow \mathbb{C}$, satisfying being compatible with the compositions.

As we anticipated, following J. Lurie we will define the category of derived \mathbb{C} -analytic spaces $\text{dAn}_{\mathbb{C}}$ as a full subcategory of $\text{Top}(\mathcal{T}_{\text{an}})$ (see Definition 1.3 for the precise formulation). We warn the reader that the category $\text{dAn}_{\mathbb{C}}$ is *different* from the category $\text{Sch}(\mathcal{T}_{\text{an}})$ we introduced before. A careful comparison of the two has been carried out by J. Lurie in [11, Corollary 12.22, Proposition 12.23].

The last important idea of [12] that plays a major role in this paper is the notion of *relative spectrum*, which is used in [11, Remark 12.26] in order to define the analytification functor. The starting point is the universal property of the classical analytification functor described by Grothendieck in [9, Exposé XII]. Recall that if X is a scheme locally of finite presentation over \mathbb{C} , then an analytification of X is the given of a \mathbb{C} -analytic space X^{an} together with a morphism of *locally ringed spaces* $X^{\text{an}} \rightarrow X$ inducing isomorphisms

$$\text{Hom}_{\text{An}_{\mathbb{C}}}(Y, X^{\text{an}}) \simeq \text{Hom}_{\text{LRingSpaces}}(Y, X)$$

for every \mathbb{C} -analytic space Y . This idea generalizes directly to the derived setting. J. Lurie constructed in [12, Theorem 2.1.1] a far reaching generalization of the analytification functor, which is known as *relative spectrum* associated to a morphism of pregeometries $\varphi: \mathcal{T} \rightarrow \mathcal{T}'$. To put it simply, he proved that the natural forgetful functor

$$\mathrm{Top}(\mathcal{T}') \rightarrow \mathrm{Top}(\mathcal{T})$$

(which sends a \mathcal{T}' -structured ∞ -topos $(\mathcal{X}, \mathcal{O})$ in the \mathcal{T} -structured ∞ -topos $(\mathcal{X}, \mathcal{O} \circ \varphi)$) admits a right adjoint, denoted $\mathrm{Spec}_{\mathcal{T}}^{\mathcal{T}'}$. Furthermore he showed that this functor respects the subcategories of schemes. Then, [11, Corollary 12.22] allows to see that $\mathrm{Spec}_{\mathcal{T}_{\mathrm{\acute{e}t}}}^{\mathcal{T}_{\mathrm{an}}}$ takes the category of derived Deligne-Mumford stacks to $\mathrm{dAn}_{\mathbb{C}}$. This will therefore be the analytification functor we will be using through this paper. As its definition is abstract (and the proof of the existence rather indirect), some work is required to show that $\mathrm{Spec}_{\mathcal{T}_{\mathrm{\acute{e}t}}}^{\mathcal{T}_{\mathrm{an}}}$ enjoys the good properties everyone would expect. This is one of the main parts of this article.

The main results. We now explain the content of the current paper. In Section 1 we begin by reviewing in a less expository way the notion of pregeometry and of derived \mathbb{C} -analytic space. We take the opportunity to summarizing the main results of [11]. Next, in Section 1.2, where we introduce the notion of weak Morita equivalence of pregeometries. This is a rather technical notion that nevertheless plays an important role in the study of the properties of germs of derived \mathbb{C} -analytic spaces. The main result of this section asserts that the ∞ -categories $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}}$ are always presentable for a pregeometry \mathcal{T} and a \mathcal{T} -structure \mathcal{O} on \mathcal{X} (while $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})$ is not, in general, presentable).

In Section 2 we show an important structure theorem for $\mathcal{T}_{\mathrm{an}}$ -structures with values in spaces: namely, we show that except for the null $\mathcal{T}_{\mathrm{an}}$ -structure, they are always canonically augmented over \mathbb{C} . This is a reminiscence of the well-known fact in functional analysis asserting that a commutative \mathbb{C} -Banach algebra which is a field is in fact isomorphic to \mathbb{C} . We use this fact to give a different description of the ∞ -category of $\mathcal{T}_{\mathrm{an}}$ -structures in \mathcal{S} . It is easy to provide a model categorical presentation for this alternative description, and we exploit this fact to carry out a few basic computations that will prove useful in dealing with the analytification functor.

With Section 3 we enter in the main body of the article. The goal is to provide an functor of points description of derived \mathbb{C} -analytic spaces and use this to (partially) compare them with the higher analytic stacks introduced in [20]. To be more precise, we introduce a category of derived Stein spaces $\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}$ and we endow it with a Grothendieck topology τ and a collection of morphisms $\mathbf{P}_{\mathrm{\acute{e}t}}$. We then prove:

Theorem 3 (Proposition 3.6 and Theorem 3.7). *There exists a fully faithful functor*

$$\phi: \mathrm{dAn}_{\mathbb{C}} \rightarrow \mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau)$$

Let now $\mathrm{dAn}_{\mathbb{C}}^{\mathrm{loc}}$ be the full subcategory of $\mathrm{dAn}_{\mathbb{C}}$ spanned by those derived \mathbb{C} -analytic spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ for which \mathcal{X} is an n -localic ∞ -topos for some integer n . If $X \in \mathrm{dAn}_{\mathbb{C}}^{\mathrm{loc}}$, then $\phi(X)$ is a geometric stack with respect to the context $(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau, \mathbf{P}_{\mathrm{\acute{e}t}})$ (see [20, Definitions 2.11 and 2.15]). Vice-versa, geometric stacks for the context $(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau, \mathbf{P}_{\mathrm{\acute{e}t}})$ constitute the essential image of the restriction $\phi: \mathrm{dAn}_{\mathbb{C}}^{\mathrm{loc}} \rightarrow \mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau)$.

Remark 4. An ∞ -topos is n -localic if it is a category of sheaves on some n -category. We redirect the reader to [10, § 6.4.5] for the definition and the main properties of n -localic ∞ -topoi.

Remark 5. An analogous statement holds in the case of algebraic stacks, in which case the result can be thought as a precise comparison between the notion of derived Deligne-Mumford stack in the sense of [26] and the one introduced in [12]. This precise formulation seems to be a folklore result and it can somehow be found scattered through the DAG series of J. Lurie. Nevertheless, a concentrated proof can be found in [18, Theorem 1.7].

We end Section 3 by discussing in some details the notion of truncations of derived \mathbb{C} -analytic spaces and the basic properties enjoyed by this operation. We obtain the following comparison result:

Corollary 6 (Corollary 3.10). *Suppose $X \in \mathrm{dAn}_{\mathbb{C}}$ is a truncated derived \mathbb{C} -analytic space belonging to $\mathrm{dAn}_{\mathbb{C}}^{\mathrm{loc}}$. Then $\phi(X)$ is a higher analytic Deligne-Mumford stack in the sense of [20].*

In Section 4 we simply introduce the notion of coherent sheaf on a derived \mathbb{C} -analytic space, and we continue the comparison with [20] by proving:

Theorem 7 (Proposition 4.3). *Suppose $X \in \mathrm{dAn}_{\mathbb{C}}$ is a truncated \mathbb{C} -analytic space satisfying the same finiteness conditions of Theorem 3. Then the ∞ -category of coherent sheaves on X is equivalent to the ∞ -category of coherent sheaves on $\phi(X)$ introduced in [20].*

In Section 5, we show that we inherit from [20] a notion of proper morphism for derived Deligne-Mumford stacks. Using the comparison results Theorem 3 and Theorem 7, we obtain the first main result of this article: a derived version of Grauert's proper direct image theorem.

Theorem 8 (Proposition 5.5). *Let $f: X \rightarrow Y$ be a proper morphism of derived \mathbb{C} -analytic spaces, both satisfying the same finiteness conditions of Theorem 3. Then the derived direct image functor $Rf_*: \mathcal{O}_X\text{-Mod} \rightarrow \mathcal{O}_Y\text{-Mod}$ takes $\text{Coh}^+(X)$ to $\text{Coh}^+(Y)$.*

Section 6 is the veritable heart of the article. Here, we carry over a detailed analysis of the analytification functor introduced in [11, Remark 12.26]. We can summarize the main results of this section as follows:

Theorem 9 (Proposition 6.8 and Corollary 6.20). *Let X be a derived Deligne-Mumford stack which is locally of finite presentation over \mathbb{C} . Then:*

- (1) (flatness) *There is a natural map $X^{\text{an}} \rightarrow X$ in the category of \mathbb{E}_∞ -ringed topoi which is flat in the derived sense.*
- (2) (comparison) *If X is truncated, then $\phi(X^{\text{an}})$ coincides with the analytification of the functor of points associated to X , this analytification being understood in the sense of [20]. In particular, when X is a scheme of finite presentation over \mathbb{C} , X^{an} can be identified with the usual analytification in the sense of Serre, cf. [9, Exposé XII].*

The flatness part is perhaps the most technical part of this article. The proof relies on the computations made in Section 2. As immediate consequence of this result, we obtain a rather explicit description of the analytification of a derived Deligne-Mumford stack in terms of the analytification of its truncation (see Corollary 6.21 for a precise statement).

Flatness unlocks moreover the two GAGA theorems, which we can now prove by reducing to the analogous results proven in [20]:

Theorem 10 (Derived GAGA-1, Theorem 7.5). *Let $f: X \rightarrow Y$ be a proper morphism of derived Deligne-Mumford stacks locally of finite presentation over \mathbb{C} . Then for every $\mathcal{F} \in \text{Coh}^+(X)$ the canonical analytification map*

$$(Rf_*(\mathcal{F}))^{\text{an}} \rightarrow Rf_*^{\text{an}}(\mathcal{F}^{\text{an}})$$

is an equivalence.

Theorem 11 (Derived GAGA-2, Theorem 7.7). *Let X be a proper Deligne-Mumford stack locally of finite presentation over \mathbb{C} . Then the analytification functor induces an equivalence of ∞ -categories*

$$\text{Coh}(X) \rightarrow \text{Coh}(X^{\text{an}})$$

Remark 12. In [20] the second GAGA theorem is stated only for the categories Coh^b . However, the hard part of the proof is to deal with the case of sheaves in

the heart $\mathrm{Coh}^\heartsuit(X)$. When [20] appeared, T. Y. Yu and I weren't aware that the same proof could also yield this stronger result.

We conclude the article with the short Section 8, where we explain how the main results of this article can be extended to derived Artin stacks.

Future work. This paper is part of my ongoing Ph.D. thesis at the university of Paris Diderot. It is part of a larger program of exploration of derived analytic geometry. In some sense, it is also a natural continuation of [20].

It will soon be followed by two related papers, [18, 19]. In [18] we carry out a similar analysis to the one of Section 3 in the setting of derived algebraic geometry. In other words, we set a precise comparison between the notion of derived Deligne-Mumford stack of [26] and the one of [12]. This is certainly a folklore result which is nevertheless difficult to find in the literature. As the proof is subtler than the one we carry over in Section 3, we feel like it deserves to be discussed at length and separately.

On the other side, in [19] we will deal more extensively with the notion of (coherent) sheaf of modules over an analytic space. There are in fact at least two reasonable definitions for this category, and both are important to have a reasonably complete theory. A large part of [19] will be devoted to the discussion of a comparison between these two notions. We plan to apply this theory to provide a workable theory of square-zero extensions and Postnikov truncations in the setting of derived \mathbb{C} -analytic geometry.

About longer term projects, we plan to concentrate on a couple of very natural questions that this article leaves somehow open. Namely, the proof of the first GAGA theorem Theorem 7.5 can be easily extended (both in the derived and in the underived setting) to the *unbounded* category of coherent sheaves under the additional requirement for the maps f and f^{an} to be of finite cohomological dimension. On the algebraic side one can reason by noetherian induction using the generic flatness lemma [20, Lemma 8.5] to show that properness implies bounded cohomological dimension. On the analytic side the picture seems slightly more complicated. As it seems an interesting question on its own, we will investigate it further.

Another question left aside both from this paper and from [20] is a version of both GAGA theorems relative to a \mathbb{C} -analytic base (note that instead the underived rigid analytic version has been dealt with in [20]). This problem is interesting as it is often needed in the practical problems of computing the analytifications of given geometric stacks. We will therefore continue to pursue the matter.

On a even longer term program, and for sake of an ongoing joint project with T. Y. Yu we are currently investigating the properties of the analytic cotangent

complex and its relation with a version of Artin representability theorem for derived \mathbb{C} -analytic spaces.

Finally, we also plan to apply the derived GAGA theorems to the study of some examples of derived non abelian mixed hodge structures, following a recent proposal of Simpson, Toën, Vaquié and Vezzosi and their kind suggestion.

Last but not least, Jacob Lurie has very recently informed us that some of the results in this paper will also appear in a forthcoming draft of his new book. The two authors worked on this topics independently, so the two final versions will probably differ. As an example, our GAGA Theorems 7.5 and 7.7 are planned to appear in such a first draft under the additional assumption that the involved stacks are actually spectral algebraic spaces. As J. Lurie informed us, he planned to include our version for derived Deligne-Mumford stacks in a later version of the aforementioned book.

Conventions. Throughout this paper we will work freely with the notion of $(\infty, 1)$ -category. We will refer to such objects simply as ∞ -categories. Sometimes, it will be necessary to consider $(n, 1)$ -categories. We refer to [10, §2.3.4] for the basic theory of such objects. In the article, we will denote them simply by n -categories (no confusion will arise because there will be no need of considering objects such as (∞, n) -categories throughout this paper).

For reasons of practical convenience, we chose to work within the framework of [10] and of [15]. When citing from these sources, we will suppress the words Definition, Lemma, Proposition etc. The notation \mathcal{S} will be reserved to the ∞ -category of spaces. In [10, 6.3.1.5] two categories of ∞ -topoi are introduced, $\mathcal{L}\mathcal{T}\mathcal{o}\mathcal{p}$ and $\mathcal{R}\mathcal{T}\mathcal{o}\mathcal{p}$. If not specified otherwise, we will denote by $\mathcal{T}\mathcal{o}\mathcal{p}$ the ∞ -category $\mathcal{R}\mathcal{T}\mathcal{o}\mathcal{p}$.

We will denote by $\mathcal{C}\mathcal{A}\mathcal{l}\mathcal{g}_{\mathbb{C}}$ the ∞ -category of connective $\mathbb{H}\mathbb{C}$ -algebras. Equivalently, this can be identified with the underlying ∞ -category of simplicial \mathbb{C} -algebras. Since we work within the derived framework most of the time, we prefer to reserve the notation $B \otimes_A C$ for the derived tensor product. Whenever the underived one is needed, we will denote it by $\mathrm{Tor}_0^A(B, C)$.

Acknowledgments. I tried to make my intellectual debt to J. Lurie and his monumental work as evident as possible since the very introduction, and I would like to emphasize it once more here. I am deeply thankful to my advisor Gabriele Vezzosi for suggesting me this very interesting research topic, for his kindness and generosity in suggesting many possible further developments. I would like to express my gratitude toward Carlos Simpson, who, directly and indirectly, encouraged me in moving on during this project. Finally, I am very grateful to V. Melani, M. Robalo, G. Vezzosi and T. Y. Yu for countless many stimulating discussions and for very helpful comments that helped me while writing this paper. I would also like

to thank V. F. Zenobi for introducing me to the idea of the holomorphic functional calculus.

1. THE SETUP

1.1. The \mathbb{C} -analytic pregeometry. This section is mainly for the convenience of the reader. We review the basic definitions and results of [11].

Definition 1.1 ([12, Def. 1.2.1]). Let \mathcal{C} be an ∞ -category. An *admissibility structure* on \mathcal{C} consists of the following data:

- (1) a subcategory $\mathcal{C}^{\text{ad}} \subset \mathcal{C}$ containing every object of \mathcal{C} . Morphisms in \mathcal{C}^{ad} will be referred to as *admissible* morphisms.
- (2) a Grothendieck topology on \mathcal{C} which is generated by admissible morphisms in the following sense: every covering sieve $\mathcal{C}_{/X}^{(0)} \subset \mathcal{C}_{/X}$ contains a covering sieve which is generated by admissible morphisms $\{U_\alpha \rightarrow X\}$.

These data are required to satisfy the following compatibilities:

- (1) whenever $f: U \rightarrow X$ is an admissible morphism and $g: X' \rightarrow X$ is any morphism, the pullback

$$\begin{array}{ccc} U' & \xrightarrow{f'} & X' \\ \downarrow & & \downarrow g \\ U & \xrightarrow{f} & X \end{array}$$

exists and f' is again an admissible morphism.

- (2) Given a commutative triangle in \mathcal{C}

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow h & \swarrow g \\ & Z & \end{array}$$

such that g and h are admissible, the same goes for f .

- (3) Admissible morphisms are stable under retracts.

Definition 1.2 ([12, Def. 3.1.1]). A pregeometry \mathcal{T} consists of an ∞ -category with finite products and an admissibility structure on it.

We let \mathcal{T}_{an} be the pregeometry defined as follows: the underlying ∞ -category is the (nerve of) the category of open subsets of \mathbb{C}^n . We say that a morphism is admissible if it is an open immersion and we endow \mathcal{T}_{an} with the analytic (Grothendieck) topology. Following [11] we introduce the notion of derived \mathbb{C} -analytic space:

Definition 1.3 ([11, Def. 12.3]). A derived \mathbb{C} -analytic space is a \mathcal{T}_{an} -structured ∞ -topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that

- (1) there exists a covering U_i of \mathcal{X} such that $\mathcal{X}_{/U_i}$ is the ∞ -topos of a topological space X_i .
- (2) The pair $(X_i, \pi_0(\mathcal{O}_{\mathcal{X}}^{\text{alg}}|_{U_i}))$ is a classical \mathbb{C} -analytic space.
- (3) For every i , the sheaves $\pi_i(\mathcal{O}_{\mathcal{X}}^{\text{alg}}|_{U_i})$ are coherent as $\pi_0(\mathcal{O}_{\mathcal{X}}^{\text{alg}}|_{U_i})$ -modules.

We will denote by $\text{dAn}_{\mathbb{C}}$ the ∞ -category of derived \mathbb{C} -analytic spaces.

We summarize the basic results about $\text{dAn}_{\mathbb{C}}$ in the following proposition:

Proposition 1.4. (1) *The category $\text{dAn}_{\mathbb{C}}$ has pullbacks, and the forgetful functor $\text{dAn}_{\mathbb{C}} \rightarrow \mathcal{T}_{\text{op}}$ commutes with them.*
 (2) *There is a canonical functor $\Phi: \text{An}_{\mathbb{C}} \rightarrow \text{dAn}_{\mathbb{C}}$ going from classical \mathbb{C} -analytic spaces to derived \mathbb{C} -analytic spaces which is fully faithful. Moreover, it commutes with pullbacks along local biholomorphisms.*
 (3) *The natural forgetful functor $\text{dAn}_{\mathbb{C}} \rightarrow \mathcal{T}_{\text{op}}(\mathcal{T}_{\text{ét}})$ commutes with pullbacks along closed immersions.*

Proof. These results are obtained in [11, §12]. The functor Φ is defined as follows: if X is a \mathbb{C} -analytic space, $\Phi(X) = (\text{Sh}(X), \mathcal{O}_{\mathcal{X}})$, where $\text{Sh}(X)$ is the ∞ -topos of (a priori non-hypercomplete) sheaves on X , and $\mathcal{O}_{\mathcal{X}}$ is the functor

$$\mathcal{O}_{\mathcal{X}}: \mathcal{T}_{\text{an}} \rightarrow \text{Sh}(X)$$

defined by

$$\mathcal{O}_{\mathcal{X}}(U) := \text{Hom}(U, -): \text{Opens}(X) \rightarrow \mathcal{S}$$

where $\text{Hom}(U, V)$ for V an open in X denotes the set of holomorphic maps from U to V . In particular, we see that $\mathcal{O}_{\mathcal{X}}$ is 0-truncated. \square

Remark 1.5. The functor $\Phi: \text{An}_{\mathbb{C}} \rightarrow \text{dAn}_{\mathbb{C}}$ factorizes by construction through the full subcategory of derived \mathbb{C} -analytic spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that $\mathcal{O}_{\mathcal{X}}$ is 0-truncated and \mathcal{X} is 0-localic. It can be shown that every such derived \mathbb{C} -analytic space lies in the essential image of Φ .

To ease future reference, we collect here also the definitions of étale morphism and closed immersion of derived \mathbb{C} -analytic spaces.

Definition 1.6. Let $f: (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \rightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of derived \mathbb{C} -analytic spaces. We will say that:

- (1) ([12, Definition 2.3.1]) f is *étale* if the underlying geometric morphism of ∞ -topoi $f^{-1}: \mathcal{Y} \rightleftarrows \mathcal{X}: f_*$ is étale in the sense of [10, § 6.3.5] and the induced morphism $f^{-1}\mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{O}_{\mathcal{X}}$ is an equivalence.

- (2) ([11, Definition 1.1]) f is a *closed immersion* if the underlying geometric morphism of ∞ -topoi $f^{-1}: \mathcal{Y} \rightleftarrows \mathcal{X}: f_*$ is a closed immersion in the sense of [10, 7.3.2.7] and the induced morphism $f^{-1}\mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{O}_{\mathcal{X}}$ is an effective epimorphism.

Let $\mathcal{T}_{\text{disc}} = \mathcal{T}_{\text{disc}}(\mathbb{C})$ be the pregeometry discussed in the Introduction, Example 1. There is a natural morphism of pregeometries $\mathcal{T}_{\text{disc}} \rightarrow \mathcal{T}_{\text{an}}$ which induces a forgetful functor

$$\text{Top}(\mathcal{T}_{\text{an}}) \rightarrow \text{Top}(\mathcal{T}_{\text{disc}})$$

Accordingly to the notations introduced in [11], if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \text{Top}(\mathcal{T}_{\text{an}})$, we will denote by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{alg}})$ its image under this functor. Observe that $\mathcal{O}_{\mathcal{X}}^{\text{alg}}$ can be identified with a sheaf of connective HC -algebras on \mathcal{X} .

1.2. Weak Morita equivalences of pregeometries. Let \mathcal{T} be a pregeometry and let \mathcal{X} be an ∞ -topos. In general, the ∞ -category of \mathcal{T} -structures on \mathcal{X} is not a presentable ∞ -category. The problem is that $\text{Str}_{\mathcal{T}}(\mathcal{X})$ and $\text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})$ are not cocomplete (see [12, Prop. 1.5.1] for a more detailed discussion), and the ultimate reason for this is to be found in the compatibilities that the objects of $\text{Str}_{\mathcal{T}}(\mathcal{X})$ are required to have with the Grothendieck topology of \mathcal{T} . This could be an issue in developing derived \mathbb{C} -analytic geometry. However, it is still true in general that, whenever $\mathcal{O} \in \text{Str}_{\mathcal{T}}(\mathcal{X})$ is a \mathcal{T} -structure, the overcategory $\text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}}$ is a presentable ∞ -category. The goal of this subsection is to provide a proof of this statement. In the economy of the present work, the relevance of this section is mainly for its application to the structure of local analytic rings, that will be discussed in the next section.

We begin with a definition:

Definition 1.7. Let $\varphi: \mathcal{T}' \rightarrow \mathcal{T}$ be a morphism of pregeometries. We will say that φ is a *weak Morita equivalence* if for every ∞ -topos \mathcal{X} and every \mathcal{T} -structure $\mathcal{O} \in \text{Str}_{\mathcal{T}}(\mathcal{X})$ the restriction functor

$$\text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}} \rightarrow \text{Str}_{\mathcal{T}'}^{\text{loc}}(\mathcal{X})_{/\mathcal{O} \circ \varphi}$$

is an equivalence of ∞ -categories.

Remark 1.8. In [12, §3.2] J. Lurie introduced the notion of Morita equivalence of pregeometries in order to understand which modifications of the admissibility structure on a pregeometry \mathcal{T} gives rise to the same ∞ -category of \mathcal{T} -structured topoi. The notion we introduced above is quite different in the spirit. It is in fact meant to be an intermediate step between Morita equivalences and the other notion, not yet appeared, of *stable Morita equivalences*. The latter is of great importance for the theory of modules for analytic structures, and we will come back on the

subject in [19]. Some of the results collected here which do not find an immediate application are stated for sake of future reference.

Lemma 1.9. *Let \mathcal{T} be a pregeometry and let \mathcal{X} be an ∞ -topos. For every $\mathcal{O} \in \text{Str}_{\mathcal{T}}(\mathcal{X})$ the canonical functor*

$$j: \text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}} \rightarrow \text{Str}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}}$$

is fully faithful.

Proof. The functor $\text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}} \rightarrow \text{Str}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}}$ is faithful, and therefore j is faithful as well. Let $\mathcal{A}, \mathcal{B} \in \text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}}$ and denote by $p: \mathcal{A} \rightarrow \mathcal{O}$, $q: \mathcal{B} \rightarrow \mathcal{O}$ the structural morphisms. If we show that every morphism $\alpha: \mathcal{A} \rightarrow \mathcal{B}$ is a local morphism, the fullness of j will follow at once. Let $f: U \rightarrow V$ be an admissible morphism in \mathcal{T} and consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{A}(U) & \xrightarrow{\alpha_U} & \mathcal{B}(U) & \xrightarrow{q_U} & \mathcal{O}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}(V) & \xrightarrow{\alpha_V} & \mathcal{B}(V) & \xrightarrow{q_V} & \mathcal{O}(V) \end{array}$$

The horizontal composites are equivalent to p_U and p_V respectively. Since p and q are local morphisms, we see that the right square as well as the outer one are pullback. It follows that the left square is a pullback too, completing the proof. \square

Lemma 1.10. *Let \mathcal{C} be an ∞ -category and let X be an object in \mathcal{C} . There is a faithful functor*

$$\mathcal{C}_{/X} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$$

whose essential image is given by those arrows with target equivalent to X . Moreover, a morphism in $\text{Fun}(\Delta^1, \mathcal{C})$ belongs to $\mathcal{C}_{/X}$ if and only its restriction to the target is equivalent to the identity of X .

Proof. The simplicial model for $\mathcal{C}_{/X}$ of [10, 1.2.9.2] provides us with the desired functor. Moreover, $\mathcal{C}_{/X}$ can be exhibited as the (homotopy) pullback diagram

$$\begin{array}{ccc} \mathcal{C}_{/X} & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow \text{ev}_1 \\ \Delta^0 & \xrightarrow{X} & \mathcal{C} \end{array}$$

from which the lemma follows. \square

Proposition 1.11. *Let \mathcal{T} be a pregeometry and let \mathcal{X} be an ∞ -topos. Let $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ be a \mathcal{T} -structure. There exists a faithful functor*

$$\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}} \rightarrow \mathrm{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X}) \quad (1.1)$$

whose essential image consists of those functors $F: \mathcal{T} \times \Delta^1 \rightarrow \mathcal{X}$ satisfying the following conditions:

- (1) *the restriction $F_1 := F|_{\mathcal{T} \times \{1\}}$ is equivalent to \mathcal{O} ;*
- (2) *the restriction $F_0 := F|_{\mathcal{T} \times \{0\}}$ commutes with products;*
- (3) *for every admissible morphism $f: U \rightarrow V$ in \mathcal{T} the induced square*

$$\begin{array}{ccc} F_0(U) & \longrightarrow & F_0(V) \\ \downarrow & & \downarrow \\ F_1(U) & \longrightarrow & F_1(V) \end{array}$$

is a pullback square.

Moreover, let $F, G \in \mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}}$ and let $\alpha: F \rightarrow G$ be a morphism between them in $\mathrm{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X})$. Then α belongs to $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}}$ if and only if the restriction $\alpha|_{\mathcal{T} \times 0}$ is equivalent to the identity of \mathcal{O} .

Proof. We can factor the functor (1.1) as

$$\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}} \xrightarrow{u} \mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}} \xrightarrow{v} \mathrm{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X}).$$

Lemma 1.9 shows that u is fully faithful. The first statement and the last one follow therefore from Lemma 1.10. It is moreover clear that all the objects in the essential image of (1.1) satisfy conditions (1) to (3). Let $F: \mathcal{T} \times \Delta^1 \rightarrow \mathcal{X}$ be a functor satisfying these conditions. Using (1) and (3) we see that this functor determines a local morphism $F_0 \rightarrow \mathcal{O}$ in $\mathrm{Fun}(\mathcal{T}, \mathcal{X})$. To conclude, we only need to show that $F_0 \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$. Since F_0 commutes with products in virtue of condition (2), we only need to check that F_0 commutes with admissible pullbacks and takes coverings in \mathcal{T} to effective epimorphisms. Let

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow f' & & \downarrow f \\ V' & \longrightarrow & V \end{array}$$

be a pullback square in \mathcal{T} , where $f: U \rightarrow V$ is an admissible morphism. Consider the commutative diagram

$$\begin{array}{ccccc} F_0(U') & \longrightarrow & F_0(U) & \longrightarrow & \mathcal{O}(U) \\ \downarrow & & \downarrow & & \downarrow \\ F_0(V') & \longrightarrow & F_0(V) & \longrightarrow & \mathcal{O}(V) \end{array}$$

and observe that the right square is a pullback because f is admissible. Moreover, the outer square can be also factored as

$$\begin{array}{ccccc} F_0(U') & \longrightarrow & \mathcal{O}(U') & \longrightarrow & \mathcal{O}(U) \\ \downarrow & & \downarrow & & \downarrow \\ F_0(V') & \longrightarrow & \mathcal{O}(V') & \longrightarrow & \mathcal{O}(V) \end{array}$$

The left square is a pullback because $\mathcal{O} \in \text{Str}_{\mathcal{T}}(\mathcal{X})$, and the left square is a pullback because $f': U' \rightarrow V'$ is an admissible pullback. It follows that F_0 preserves admissible pullbacks.

Now let $\{U_i \rightarrow U\}$ be an admissible covering in \mathcal{T} . Since \mathcal{X} is an ∞ -topos, we see that there is a pullback square

$$\begin{array}{ccc} \coprod F_0(U_i) & \longrightarrow & F_0(U) \\ \downarrow & & \downarrow \\ \coprod \mathcal{O}(U_i) & \longrightarrow & \mathcal{O}(U) \end{array}$$

Since the bottom line is an effective epimorphism, the result now follows from the stability of effective epimorphisms under pullbacks. \square

Proposition 1.11 allows to produce several examples of weak Morita equivalences.

Definition 1.12. Let \mathcal{T} be a pregeometry. The *associated discrete pregeometry* \mathcal{T}_d is the discrete pregeometry having the same underlying ∞ -category of \mathcal{T} . The *associated semi-discrete pregeometry* \mathcal{T}_{sd} is the pregeometry having the same underlying ∞ -category and the same admissible morphisms of \mathcal{T} , but discrete Grothendieck topology.

Observe that we have morphisms of pregeometries $\mathcal{T}_d \rightarrow \mathcal{T}_{sd} \rightarrow \mathcal{T}$.

Corollary 1.13. *Let \mathcal{T} be a pregeometry. The morphism $\mathcal{T}_{sd} \rightarrow \mathcal{T}$ is a weak Morita equivalence.*

Proof. Let \mathcal{X} be an ∞ -topos and let $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$. Denote by \mathcal{O}' the restriction of \mathcal{O} along $\mathcal{T}_{\mathrm{sd}} \rightarrow \mathcal{T}$. Proposition 1.11 allows to identify both $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}}$ and $\mathrm{Str}_{\mathcal{T}_{\mathrm{sd}}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}'}$ with the same subcategory of $\mathrm{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X})$. Therefore, they are equivalent. \square

Lemma 1.14. *Let \mathcal{T} be a pregeometry in which the Grothendieck topology is discrete. Then for every ∞ -topos \mathcal{X} , the ∞ -category $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ is presentable.*

Proof. This is a direct consequence of [10, Lemmas 5.5.4.18, 5.5.4.19]. \square

Proposition 1.15. *Let \mathcal{T} be a pregeometry and let \mathcal{X} be an ∞ -topos. For every $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ the ∞ -category $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}}$ is presentable.*

Proof. In virtue of Corollary 1.13 we can assume that the Grothendieck topology on \mathcal{T} is discrete. In this case, Lemma 1.14 shows that $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ is a presentable ∞ -category. [12, Theorem 1.3.1] shows that there exists a factorization system $(S_L^{\mathcal{X}}, S_R^{\mathcal{X}})$ on $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ such that $S_R^{\mathcal{X}}$ is the collection of morphisms in $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})$.

Let \mathcal{D} be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Str}_{\mathcal{T}}(\mathcal{X}))$ spanned by the elements of $S_R^{\mathcal{X}}$. [10, Lemma 5.2.8.19] shows that \mathcal{D} is a localization of $\mathrm{Fun}(\Delta^1, \mathrm{Str}_{\mathcal{T}}(\mathcal{X}))$. Since filtered colimits commutes with pullbacks, we see that \mathcal{D} is closed under filtered colimits in $\mathrm{Fun}(\Delta^1, \mathrm{Str}_{\mathcal{T}}(\mathcal{X}))$. In other words, \mathcal{D} is an accessible localization of the latter category. Since $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ is a presentable ∞ -category, [10, Proposition 5.5.1.2] shows that \mathcal{D} is presentable as well. Observe now that $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}}$ fits into a pullback square

$$\begin{array}{ccc} \mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \mathrm{ev}_1 \\ \{\mathcal{O}\} & \longrightarrow & \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \end{array}$$

[10, Theorem 5.5.3.18] shows that $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}}$ is presentable, thus completing the proof. \square

For future convenience, we record the following easy consequence:

Corollary 1.16. *Let \mathcal{T} be a pregeometry and let \mathcal{X} be an ∞ -topos. For every $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ the category of \mathcal{O} -modules $\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}})$ is a presentable ∞ -category.*

Proof. This follows from Proposition 1.15 and [15, Proposition 1.4.4.4.(1)]. \square

Proposition 1.17. *Let \mathcal{T} be a pregeometry and let \mathcal{X} be an ∞ -topos. Let $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ be a \mathcal{T} -structure. The functor $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}} \rightarrow \mathrm{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X})$ creates connected limits and sifted colimits.*

Proof. Let K be a simplicial set and let $F: K \rightarrow \mathrm{Fun}(\mathcal{T} \times \Delta, \mathcal{X})$ be a functor factorizing through $\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}}$. Let $\tilde{F}: K^{\triangleright} \rightarrow \mathrm{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X})$ be a limit diagram

and set $G := F(v_0)$, where v_0 denotes the initial vertex of K^\triangleleft . G_0 commutes with product and G_1 is the limit of the constant diagram associated to \mathcal{O} . Since sifted colimits are connected, we see that G_1 is equivalent to \mathcal{O} . Moreover, if $f: U \rightarrow V$ is an admissible morphism in \mathcal{T} , the diagram

$$\begin{array}{ccc} G_0(U) & \longrightarrow & G_0(V) \\ \downarrow & & \downarrow \\ G_1(U) & \longrightarrow & G_1(V) \end{array}$$

is the limit of pullback diagrams and it is therefore a pullback diagram by itself.

Let now K be a sifted simplicial set and let $F: K \rightarrow \text{Fun}(\mathcal{T} \times \Delta, \mathcal{X})$ be a functor factorizing through $\text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}}$. Let $\tilde{F}: K^\triangleleft \rightarrow \text{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X})$ be a colimit diagram and set $G := F(v_\infty)$, where v_∞ denotes the final object of K^\triangleleft . Since K is sifted, we see that G_0 commutes with products. As before, we see that G_1 is equivalent to \mathcal{O} . Let $f: U \rightarrow V$ be an admissible morphism in \mathcal{T} . Since \mathcal{X} is an ∞ -topos, we see that

$$\begin{aligned} G_1(U) \times_{G_1(V)} G_0(V) &\simeq \mathcal{O}(U) \times_{\mathcal{O}(V)} G_0(V) \\ &\simeq \mathcal{O}(U) \times_{\mathcal{O}(V)} \text{colim}_K(F_0(V)) \\ &\simeq \text{colim}_K(\mathcal{O}(U) \times_{\mathcal{O}(V)} F_0(V)) \simeq \text{colim}_K F_0(U) \simeq G_0(U) \end{aligned}$$

We therefore conclude that G belongs to the $\text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}}$. \square

Corollary 1.18. *Let $\varphi: \mathcal{T}' \rightarrow \mathcal{T}$ be a morphism of pregeometries. Let $\mathcal{O} \in \text{Str}_{\mathcal{T}}(\mathcal{X})$ be a \mathcal{T} -structure. The restriction functor*

$$\bar{\Phi}: \text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}} \rightarrow \text{Str}_{\mathcal{T}'}^{\text{loc}}(\mathcal{X})_{/\mathcal{O} \circ \varphi}$$

commutes with limits and sifted colimits.

Proof. Observe that the relevant functor takes the final object to the final object by the very construction. It will be therefore sufficient to show that it commutes with connected limits and sifted colimits. Consider the commutative diagram

$$\begin{array}{ccc} \text{Str}_{\mathcal{T}}^{\text{loc}}(\mathcal{X})_{/\mathcal{O}} & \longrightarrow & \text{Str}_{\mathcal{T}'}^{\text{loc}}(\mathcal{X})_{/\mathcal{O} \circ \varphi} \\ \downarrow & & \downarrow \\ \text{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X}) & \longrightarrow & \text{Fun}(\mathcal{T}' \times \Delta^1, \mathcal{X}) \end{array}$$

Proposition 1.17 shows that the vertical morphisms creates both connected limits and sifted colimits, while $\text{Fun}(\mathcal{T} \times \Delta^1, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{T}' \times \Delta^1, \mathcal{X})$ preserves all (small) limits and (small) colimits. The proof is therefore complete. \square

Corollary 1.19. *Let $\varphi: \mathcal{T}' \rightarrow \mathcal{T}$ be a morphism of pregeometries. Let $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ be a \mathcal{T} -structure. The restriction functor*

$$\mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O}} \rightarrow \mathrm{Str}_{\mathcal{T}}^{\mathrm{loc}}(\mathcal{X})_{/\mathcal{O} \circ \varphi}$$

is a right adjoint.

2. LOCAL ANALYTIC RINGS

The goal of this section is to analyze in detail the category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}^{\mathrm{loc}}(\mathcal{S})$. We introduce a special notation:

Definition 2.1. The ∞ -category of *local analytic rings* is the ∞ -category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}^{\mathrm{loc}}(\mathcal{S})$. We will denote it by $\mathrm{AnRing}_{\mathbb{C}}$.

2.1. Functional spectrum. Let us fix some notations. We will denote by \mathcal{H}_0 the $\mathcal{T}_{\mathrm{an}}$ -structure defined by

$$\mathcal{H}_0(U) = \mathrm{Hom}_{\mathcal{T}_{\mathrm{an}}}(\{*\}, U) \simeq U \in \mathcal{S}$$

where U is thought as a simple (discrete) set. Observe that \mathcal{H}_0 is indeed a $\mathcal{T}_{\mathrm{an}}$ -structure and that it is moreover discrete. If $\mathcal{O} \in \mathrm{AnRing}_{\mathbb{C}}$, then we see that, by the Yoneda lemma, one has

$$\mathrm{Map}_{\mathrm{AnRing}_{\mathbb{C}}}(\mathcal{H}_0, F) \simeq F(\{*\}) \simeq \{*\}$$

In particular, \mathcal{H}_0 is an initial object in $\mathrm{AnRing}_{\mathbb{C}}$. On the other side the category $\mathrm{AnRing}_{\mathbb{C}}$ has a final object, which we will denote by 0 . We will refer to it as the *null analytic ring*. It is the constant functor associated to $\{*\} \in \mathcal{S}$.

The main goal of this section is to prove the following result:

Theorem 2.2. *Let $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})'$ be the full subcategory of $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})$ spanned by every object but the null analytic ring. Then \mathcal{H}_0 is a final object in $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})'$. Moreover, for every $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})'$ the canonical map $\mathcal{O} \rightarrow \mathcal{H}_0$ is a local transformation.*

Corollary 2.3. *One has $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})' \simeq \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})_{/\mathcal{H}_0} \simeq \mathrm{AnRing}_{\mathbb{C}/\mathcal{H}_0}$. In particular, all these categories are presentable.*

Proof. The two equivalences follow directly from the theorem. The last assertion is a consequence of Proposition 1.15. \square

Remark 2.4. We also see that the above corollary, together with Lemma 1.9, implies that $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}^{\mathrm{loc}}(\mathcal{S}) \simeq \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})$.

Definition 2.5. We will refer to the ∞ -category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}^{\mathrm{loc}}(\mathcal{S})_{/\mathcal{H}_0}$ as the ∞ -category of *local analytic rings*. We will denote it $\mathrm{AnRing}_{\mathbb{C}}^{\mathrm{loc}}$.

Observe that since \mathcal{H}_0 is 0-truncated, any map $\mathcal{O} \rightarrow \mathcal{H}_0$ factors uniquely as $\mathcal{O} \rightarrow \tau_{\leq 0}\mathcal{O} \rightarrow \mathcal{H}_0$. Observe also that since \mathcal{T}_{an} is compatible with n -truncations (see [12, Definition 3.3.2] for the definition and [11, Proposition 11.4] for a proof), $\tau_{\leq 0}\mathcal{O}$ is a \mathcal{T}_{an} -structure in \mathcal{S} , and the morphism $\mathcal{O} \rightarrow \tau_{\leq 0}\mathcal{O}$ is a local morphism. Therefore, to prove Theorem 2.2 it is sufficient to deal with *discrete* local analytic rings. We will denote by $\text{AnRing}_{\mathbb{C}}^0$ the full subcategory of $\text{AnRing}_{\mathbb{C}}$ spanned by discrete objects. The above considerations show that the first statement of Theorem 2.2 is in fact equivalent to prove the following key result:

Proposition 2.6. *Let $\mathcal{O} \in \text{AnRing}_{\mathbb{C}}^0$ and suppose $\mathcal{O} \neq 0$. Then \mathcal{O}^{alg} is a local ring with residue field \mathbb{C} .*

Remark 2.7. Observe that this result would be false if we replaced \mathcal{T}_{an} with \mathcal{T}_{Zar} .

We will need several lemmas.

Lemma 2.8. *Let $\mathcal{O} \in \text{AnRing}_{\mathbb{C}}^{\text{loc}}$ and suppose $\mathcal{O} \neq 0$. Then $\mathcal{O}(\emptyset) = \emptyset$.*

Proof. Since we have an admissible morphism $\emptyset \rightarrow \{*\}$, we see that $\mathcal{O}(\emptyset) \rightarrow \mathcal{O}(\{*\})$ is a monomorphism. Therefore $\mathcal{O}(\emptyset)$ has cardinality at most 1. Suppose by contradiction that $\mathcal{O}(\emptyset)$ has one element. Then, consider the commutative triangle

$$\begin{array}{ccc} \emptyset & & \\ \downarrow & \searrow & \\ \mathbb{C} & \xrightarrow{t_a} & \mathbb{C}, \end{array}$$

where t_a denotes the translation by the element $a \in \mathbb{C}$. Applying \mathcal{O} we would get another commutative triangle, and the morphism $\mathcal{O}(\emptyset) \rightarrow \mathcal{O}(\mathbb{C})$ would select an element $f \in \mathcal{O}(\mathbb{C})$ such that $f + a = f$. This is impossible as soon as $a \neq 0$. Therefore $\mathcal{O}(\emptyset) = \emptyset$. \square

Let $\mathcal{O} \in \text{AnRing}_{\mathbb{C}}^0$ be a non-null discrete analytic ring and let $f \in \mathcal{O}^{\text{alg}}$. Define the spectrum of f as follows:

$$\sigma_{\mathcal{O}}(f) = \sigma(f) := \{a \in \mathbb{C} \mid f - a \notin (\mathcal{O}^{\text{alg}})^{\times}\}$$

Lemma 2.9. *Let $f \in \mathcal{O}^{\text{alg}}$. Then $a \notin \sigma(f)$ if and only if $f \in \mathcal{O}(\mathbb{C}_a^*)$, where $\mathbb{C}_a^* := \mathbb{C} \setminus \{a\}$.*

Proof. When $a = 0$, we will simply write \mathbb{C}^* instead of \mathbb{C}_0 . If $f \in \mathcal{O}(\mathbb{C}_a^*)$ then $f - a \in \mathcal{O}(\mathbb{C}^*)$ and therefore $f - a \in (\mathcal{O}^{\text{alg}})^{\times}$. Conversely, suppose that $f - a$ is

invertible. Consider the pullback square

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}^* & \longrightarrow & \mathbb{C} \times \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}^* & \longrightarrow & \mathbb{C} \end{array}$$

The horizontal morphisms are open immersions, and therefore the induced square

$$\begin{array}{ccc} \mathcal{O}(\mathbb{C}^*) \times \mathcal{O}(\mathbb{C}^*) & \longrightarrow & \mathcal{O}(\mathbb{C}) \times \mathcal{O}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathbb{C}^*) & \longrightarrow & \mathcal{O}(\mathbb{C}) \end{array}$$

is a pullback square as well. Let g be an inverse for $f - a$. Then the pair $((f - a, g), 1)$ determines an element of the above pullback. This means that both $f - a$ and g factors through $\mathcal{O}(\mathbb{C}^*)$, completing the proof. \square

Lemma 2.10. *If $f \in \mathcal{O}^{\text{alg}}$ then $\sigma(f) \neq \emptyset$.*

Proof. Suppose by contradiction that $\sigma(f) = \emptyset$ for some $f \in \mathcal{O}^{\text{alg}}$. Then for every $a \in \mathbb{C}$, $f - a$ is invertible, that is, $f \in \mathcal{O}(\mathbb{C}_a^*)$ for every $a \in \mathbb{C}$. Fix $a \in \mathbb{C}$ and write

$$\mathbb{C}_a^* = \bigcup_{n>0} \left(\mathbb{C} \setminus \overline{D}\left(a, \frac{1}{n}\right) \right)$$

It follows that there exists $\varepsilon > 0$ such that $f \in \mathcal{O}(\mathbb{C} \setminus \overline{D}(a, \varepsilon))$. Since $\mathcal{O}(\emptyset) = \emptyset$, we conclude that $f \notin \mathcal{O}(D(a, \varepsilon))$. But then we can cover \mathbb{C} with open subsets U_i such that $f \notin \mathcal{O}(U_i)$ for every i , which is a contradiction. \square

Lemma 2.11. *Suppose that $\mathcal{O} \in \text{AnRing}_{\mathbb{C}}^0$ and let $f \in \mathcal{O}^{\text{alg}}$. Then $\sigma(f)$ has at most one point.*

Proof. Suppose that $a, b \in \sigma(f)$, with $a \neq b$. Then $f \notin \mathcal{O}(\mathbb{C}_a^*)$ and $f \notin \mathcal{O}(\mathbb{C}_b^*)$, which implies that $f \in \mathcal{O}(D(a, \varepsilon)) \cap \mathcal{O}(D(b, \eta))$ for every $\varepsilon, \eta > 0$. This is a contradiction since $\mathcal{O}(\emptyset) = \emptyset$. \square

Corollary 2.12. *Let $\mathcal{O} \in \text{AnRing}_{\mathbb{C}}^0$ be a non-null discrete analytic ring. For every $f \in \mathcal{O}^{\text{alg}}$, $\sigma(f)$ consists precisely of one point.*

At this point we obtain a well defined function of sets

$$\sigma: \mathcal{O}^{\text{alg}} \rightarrow \mathbb{C}$$

Lemma 2.13. *One has $\sigma(f + g) = \sigma(f) + \sigma(g)$ and $\sigma(fg) = \sigma(f)\sigma(g)$.*

Proof. Set $\sigma(f) = a$ and $\sigma(g) = b$. It will be sufficient to check that $a + b \in \sigma(f + g)$. Suppose that there exists $h \in \mathcal{O}^{\text{alg}}$ such that $(f + g - a - b)h = 1$. Then the ideal generated by $f - a$ and $g - b$ is the unit ideal. However, \mathcal{O} is local and both $f - a$ and $g - b$ belongs to the maximal ideal. This is therefore impossible. Similarly, suppose there exists h such that $(fg - ab)h = 1$. Then $(f - a)gh + ah(g - b) = 1$, and the proof proceeds as for the addition. \square

We can now prove Proposition 2.6:

Proof of Proposition 2.6. Let $\mathcal{O} \in \text{AnRing}_{\mathbb{C}}$ be a non-null analytic structure. We deduce from Lemma 2.13 that $\sigma: \mathcal{O}^{\text{alg}} \rightarrow \mathbb{C}$ is a morphism of rings. Lemma 2.9 shows that the kernel of σ is precisely the set of non invertible elements of \mathcal{O}^{alg} . It follows that $\ker(\sigma)$ has to be the unique maximal ideal of \mathcal{O}^{alg} . Since σ is evidently surjective, the proof is complete. \square

The next result achieves the proof of Theorem 2.2.

Proposition 2.14. *Let $\mathcal{O}_1 \in \text{AnRing}_{\mathbb{C}}^0$ be a non-null discrete local analytic ring. Then $\mathcal{O}_1^{\text{alg}}$ is canonically augmented toward \mathbb{C} and the morphism $\sigma_1: \mathcal{O}_1 \rightarrow \mathbb{C}$ comes from a local morphism $s_1: \mathcal{O}_1 \rightarrow \mathcal{H}_0$. Moreover, if \mathcal{O}_2 is any other non-null discrete local analytic ring and $f: \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is any morphism between them, one has $s_2 \circ f = s_1$.*

Proof. Let us prove that $\mathcal{O}_1 \rightarrow \mathcal{H}_0$ is a natural transformation. This is equivalent to prove that if $f \in \mathcal{O}(U)$ then $\sigma(f) \in U$. Let $a \notin U$. Then there exists $\varepsilon > 0$ such that $f \notin \mathcal{O}(D(a, \varepsilon))$, which means that $f \in \mathcal{O}_1(\mathbb{C}_a^*)$. Therefore $f - a$ is invertible, i.e. $a \notin \sigma(f)$. Therefore $\sigma(f) \in U$.

Let us now prove that $\mathcal{O}_1 \rightarrow \mathcal{H}_0$ is a local morphism. We claim that if $f \in \mathcal{O}_1(\mathbb{C})$ and $\sigma(f) \in U$, then $f \in \mathcal{O}_1(U)$. Let $a = \sigma(f)$, so that $f \notin \mathcal{O}_1(\mathbb{C}_a^*)$. Since $\mathbb{C} = \mathbb{C}_a^* \cup U$, we conclude that $f \in \mathcal{O}_1(U)$. This completes the proof.

Finally, we prove the last statement. Let $x \in \mathcal{O}_1(\mathbb{C})$, $a = \sigma_1(x)$. Then $x - a$ is non-invertible. This means that $x - a$ belongs to $\mathcal{O}_1(D(0, \varepsilon))$ for every $\varepsilon > 0$. Then $f(x - a) = f(x) - a$ belongs to $\mathcal{O}_2(D(0, \varepsilon))$ for every $\varepsilon > 0$. It follows that $\sigma_2(f(x) - a) \in D(0, \varepsilon)$ for every $\varepsilon > 0$. Therefore $\sigma_2(f(x) - a) = 0$, i.e. $\sigma_2(f(x)) = a = \sigma_1(x)$. \square

Corollary 2.15. *For any non-null local analytic ring $\mathcal{O} \in \text{AnRing}_{\mathbb{C}}$ (not necessarily discrete), the canonical morphism $\mathcal{H}_0 \rightarrow \mathcal{O}$ is a local morphism.*

2.2. Quotients of local analytic rings. This subsection and the next one are not, strictly speaking, needed for this work. Nevertheless, we chose to include them for two reasons: on one side, they help in building a 1-categorical intuition on what

kind of objects (local) analytic rings are, and on the other side the results collected here will be used in [19].

Let $A \in \text{AnRing}_{\mathbb{C}}^0$ and let $I \subset A^{\text{alg}}$ be a (proper) ideal. Observe that I is contained in the maximal ideal of A and therefore $\sigma(I) = 0$.

Construction 2.16. We will construct an analytic ring $A/I: \mathcal{T}_{\text{an}}^0 \rightarrow \text{Set}$ (we are using the results of Section 2.4). Define

$$(A/I)(p, \mathbb{C}^n) := \{\bar{a} \in (A^{\text{alg}}/I)^n \mid \sigma(a) = p\}$$

Observe that this is well defined. Indeed, if $\bar{a} = \bar{b}$, then $a - b \in I^n$ and therefore $\sigma(a - b) = 0$. Let now $\varphi: (p, \mathbb{C}^n) \rightarrow (q, \mathbb{C}^m)$ be a germ of holomorphic map. Set

$$(A/I)(\varphi)(\bar{a}) := \overline{A(\varphi)(a)}$$

This is well defined. Indeed, if $\bar{a} = \bar{b}$, then we can write $a = b + x$, where $x \in I^n$. Therefore

$$A(\varphi)(a) = A(\varphi)(b) + \sum A\left(\frac{\partial f}{\partial z_i}\right)(b)x_i + \sum A(g_{ij})(b+x)x_ix_j$$

and therefore

$$\overline{A(\varphi)(a)} = \overline{A(\varphi)(b)}$$

It follows from the construction that A/I commutes with products. In particular, we see that A/I defines a local analytic ring. Moreover, the natural projection map $p: A \rightarrow A/I$ is a morphism of local analytic rings. Observe finally that $(A/I)^{\text{alg}} = A^{\text{alg}}/I$.

Lemma 2.17. *Let $f: A \rightarrow B$ be a morphism of discrete local analytic rings. The following are equivalent:*

- (1) *the induced map $A(0, \mathbb{C}) \rightarrow B(0, \mathbb{C})$ is surjective;*
- (2) *the induced map $A(0, \mathbb{C}^n) \rightarrow B(0, \mathbb{C}^n)$ is surjective for every $n \geq 0$;*
- (3) *the induced map $A(p, \mathbb{C}^n) \rightarrow B(p, \mathbb{C}^n)$ is surjective for every n and every p .*

Proof. The implications 3. \Rightarrow 2. and 2. \Rightarrow 1. are trivial. Suppose that $A(0, \mathbb{C}) \rightarrow B(0, \mathbb{C})$ is surjective. Since both A and B commute with products, 2. follows at once. Since (p, \mathbb{C}^n) and $(0, \mathbb{C}^n)$ are isomorphic, 3. follows as well. \square

Definition 2.18. We will say that a morphism of discrete local analytic rings $f: A \rightarrow B$ is *surjective* if it satisfies the equivalent conditions of the above lemma.

Lemma 2.19. *Let $f: A \rightarrow B$ be a surjective map of discrete local analytic rings and set $I := \ker(A^{\text{alg}} \rightarrow B^{\text{alg}})$. Then for every other discrete local analytic ring C , we have*

$$\text{Hom}(B, C) = \{g: A \rightarrow C \mid g^{\text{alg}}(I) = 0\}$$

Proof. Let $g: A \rightarrow C$ be such that $g^{\text{alg}}(I) = 0$. Define $\tilde{g}: B \rightarrow C$ as follows. If $b \in B(p, \mathbb{C}^n)$, we can choose an element $a \in A(p, \mathbb{C}^n)$ such that $f(a) = b$. Define

$$\tilde{g}(b) := g(a)$$

If $a' \in A(p, \mathbb{C}^n)$ is another element satisfying $f(a') = b$, we see that $a - a' \in A(0, \mathbb{C}^n)$ and $a - a' \in I$. Therefore $f(a - a') = 0$, i.e. $f(a) = f(a')$. This shows that \tilde{g} is well defined. Let us show that \tilde{g} is a natural transformation. If $\varphi: (p, \mathbb{C}^n) \rightarrow (q, \mathbb{C}^m)$ is a germ of a holomorphic map, we can consider the diagram

$$\begin{array}{ccccc} A(p, \mathbb{C}^n) & \longrightarrow & B(p, \mathbb{C}^n) & \longrightarrow & C(p, \mathbb{C}^n) \\ \downarrow & & \downarrow & & \downarrow \\ A(q, \mathbb{C}^m) & \longrightarrow & B(q, \mathbb{C}^m) & \longrightarrow & C(q, \mathbb{C}^m) \end{array}$$

Since the maps $A(p, \mathbb{C}^n) \rightarrow B(p, \mathbb{C}^n)$ and $A(q, \mathbb{C}^m) \rightarrow B(q, \mathbb{C}^m)$ are surjective, in order to check that the square on the right commutes, it is sufficient to check that the outer rectangle does. This follows from the hypothesis that g is a natural transformation. \square

Let $A \in \text{AnRing}_{\mathbb{C}}^0$ and let $I \subset A^{\text{alg}}$ be an ideal. Then the canonical map $A \rightarrow (A/I)$ is surjective and therefore it is characterized by the universal property of the previous lemma. Vice-versa, if $A \rightarrow B$ is a surjective morphism of local analytic rings and $I := \ker(A^{\text{alg}} \rightarrow B^{\text{alg}})$, then $B \simeq A/I$.

For the next result, we will need a bit of notation. As we discussed in the introduction, analytic rings are a way of axiomatizing the holomorphic functional calculus enjoyed by commutative Banach \mathbb{C} -algebras. For this reason, we choose to denote the pushouts $B \amalg_A C$ in $\text{AnRing}_{\mathbb{C}}^0$ by $B \hat{\otimes}_A C$.

Corollary 2.20. *Suppose that $f: A \rightarrow B$ is a surjective morphism of local analytic rings. For every other morphism $g: A \rightarrow C$, we have*

$$(B \hat{\otimes}_A C)^{\text{alg}} \simeq B^{\text{alg}} \otimes_{A^{\text{alg}}} C^{\text{alg}}$$

In other words, the functor $(-)^{\text{alg}}$ commutes with pushouts in which at least one of the two maps is surjective.

Proof. Write $B = A/I$. Then

$$\text{Hom}(B \hat{\otimes}_A C, R) = \{g: C \rightarrow R \mid g^{\text{alg}}(IC^{\text{alg}}) = 0\}$$

In other words, $B \hat{\otimes}_A C = C/IC^{\text{alg}}$. It follows that $(B \hat{\otimes}_A C)^{\text{alg}} = C^{\text{alg}}/IC^{\text{alg}} = B^{\text{alg}} \otimes_{A^{\text{alg}}} C^{\text{alg}}$. \square

Remark 2.21. This result, whose counterpart for Banach algebras is very well-known, has been generalized to a great extent by J. Lurie in [11]. We can without any doubt say that this generalization is the most important idea contained there. The precise way of formulating this uses once again the language of pregeometries, and the relevant notion is that of *unramified transformation of pregeometries* (see loc. cit., Definition 10.1) The condition of being unramified is a rather simple test for a morphism of pregeometries (see loc. cit., Remark 10.2 for a simplified formulation), and yet it has the powerful consequence expressed in loc. cit., Proposition 10.3. The proof of this striking result passes through the highly technical Proposition 2.2, which is perhaps an interesting result on its own.

2.3. Analytic algebras. We will denote by $\mathbf{LRingSpaces}$ the (1-)category of locally ringed topological spaces.

Let $A_n = \mathbb{C}\{z_1, \dots, z_n\}$ be the algebra of germs of holomorphic functions around $0 \in \mathbb{C}^n$. This is an analytic algebra in the sense of Malgrange. We define an enhancement of A_n to the setting of analytic rings as follows. Define $\mathcal{H}_n: \mathcal{T}_{\text{an}} \rightarrow \mathbf{Set}$ by setting:

$$\begin{aligned} \mathcal{H}_n(U) &:= \{\text{germs of holomorphic functions } (0, \mathbb{C}^n) \rightarrow U\} \\ &= \text{Map}_{\mathbf{LRingSpaces}}((\mathcal{S}, A_n), U) \end{aligned}$$

It is clear that \mathcal{H}_n defines a functor $\mathcal{T}_{\text{an}} \rightarrow \mathbf{Set}$ and that moreover $\mathcal{H}_n(\mathbb{C}) = A_n$.

Lemma 2.22. *The functor \mathcal{H}_n is a \mathcal{T}_{an} -structure on \mathbf{Set} .*

Proof. It follows directly from the definition that \mathcal{H}_n commutes with all the limits that exist in \mathcal{T}_{an} . In particular, it commutes with products and with admissible pullbacks. Let now $\{U_i \subset U\}$ be an open cover of U . The morphisms $U_i \subset U$ are open immersions in the category of locally ringed spaces, and therefore they enjoy the following universal property: a morphism $Z \rightarrow U$ from a locally ringed space Z factors through U_i if and only if it factors topologically. Therefore, we see that for every morphism $(\mathcal{S}, A_n) \rightarrow U$, there exists an index i such that this morphism factorizes through U_i . It follows that the map $\coprod \mathcal{H}_n(U_i) \rightarrow \mathcal{H}_n(U)$ is surjective. \square

Our next goal is to characterize \mathcal{H}_n with a universal property.

Proposition 2.23. *Let $B \in \mathbf{AnRing}_{\mathbb{C}}^0$. Then*

$$\text{Hom}_{\mathbf{AnRing}_{\mathbb{C}}^0}(\mathcal{H}_n, B) = \{(b_1, \dots, b_n) \in B(\mathbb{C}) \mid \sigma(b_1) = \dots = \sigma(b_n) = 0\}$$

Proof. Suppose given a natural transformation $\varphi: \mathcal{H}_n \rightarrow B$. Let us denote by z_1, \dots, z_n the germs at 0 of the coordinate functions on \mathbb{C}^n . That is, we have

$z_1, \dots, z_n \in \mathcal{H}_n(\mathbb{C}) = \mathcal{H}_n$. Their image via φ define elements $b_1, \dots, b_n \in B$. Moreover, ?? shows that $\sigma(b_i) = \sigma(\varphi(z_i)) = \varphi(z_i) = 0$.

Conversely, suppose given elements $b_1, \dots, b_n \in B(\mathbb{C})$ such that $\sigma(b_1) = \dots = \sigma(b_n) = 0$, we can define a morphism $\varphi: \mathcal{H}_n \rightarrow B$ as follows. Let $U \in \mathcal{T}_{\text{an}}$. An element $f \in \mathcal{H}_n(U)$ can be represented by a holomorphic function $\tilde{f}: V \rightarrow U$ for some open neighborhood V of $0 \in \mathbb{C}^n$. Since the functional spectrum of b_1, \dots, b_n is zero, we see that

$$(b_1, \dots, b_n) \in B(V)$$

We therefore define

$$\varphi_U(f) := B(\tilde{f})(b_1, \dots, b_n) \in B(U)$$

If $\tilde{g}: V' \rightarrow U$ is another representation of f , then we can suppose that $V' \subset V$ and that $\tilde{g} = \tilde{f}|_{V'}$. In this case, we have a commutative triangle

$$\begin{array}{ccc} B(V') & \longrightarrow & B(V) \\ & \searrow B(\tilde{g}) & \downarrow B(\tilde{f}) \\ & & B(U) \end{array}$$

and $(b_1, \dots, b_n) \in B(V')$. This shows that the definition of $\varphi_U(f)$ does not depend on the choice of the representation of f . It is straightforward to check that φ_U defines a natural transformation. Finally, the two constructions we defined are clearly one the inverse of each other. \square

Corollary 2.24. *The coproduct of \mathcal{H}_n and \mathcal{H}_m in $\text{AnRing}_{\mathbb{C}}^0$ is \mathcal{H}_{n+m} .*

Let now I be an ideal of \mathcal{H}_n . This corresponds to a germ of \mathbb{C} -analytic space $(0, X)$, together with a map of germs $j: (0, X) \rightarrow (0, \mathbb{C}^n)$. Define a functor $\mathcal{H}_n/I: \mathcal{T}_{\text{an}} \rightarrow \text{Set}$ as follows:

$$(\mathcal{H}_n/I)(U) := \{\text{germs of holomorphic maps } (0, X) \rightarrow U\}$$

Clearly, $\mathcal{H}_n/I \in \text{AnRing}_{\mathbb{C}}^0$. Moreover, $(\mathcal{H}_n/I)(\mathbb{C}) = \mathcal{H}_n/I$. Composition with j produces a natural transformation $\pi: \mathcal{H}_n \rightarrow \mathcal{H}_n/I$.

Proposition 2.25. *Let $B \in \text{AnRing}_{\mathbb{C}}^0$. We have*

$$\text{Hom}(\mathcal{H}_n/I, B) = \{\varphi: \mathcal{H}_n \rightarrow B \mid \varphi^{\text{alg}}(I) = 0\}$$

Proof. Given $\psi: \mathcal{H}_n/I \rightarrow B$ we obtain $\varphi := \psi \circ \pi: \mathcal{H}_n \rightarrow B$, and it is clear that $\varphi^{\text{alg}}(I) = 0$. Conversely, if $\varphi: \mathcal{H}_n \rightarrow B$ is such that φ^{alg} factors through \mathcal{H}_n/I , then define $\psi: \mathcal{H}_n/I \rightarrow B$ as follows. Fix $U \in \mathcal{T}_{\text{an}}$ and let $f \in (\mathcal{H}_n/I)(U)$ be a germ of a holomorphic function $(0, X) \rightarrow U$. Using Oka principle, we can represent

f as a holomorphic map $V \rightarrow U$, where V is some open neighborhood of 0 in \mathbb{C}^n . In other words, f is the restriction of some $\tilde{f} \in \mathcal{H}_n(U)$. Set

$$\psi_U(f) := \varphi_U(\tilde{f})$$

It is clear that this definition doesn't change if we shrink V . On the other hand, suppose that $\tilde{g}: V \rightarrow U$ is another extension of the germ $f: (0, X) \rightarrow U$. Then $\tilde{f} - \tilde{g} \in \mathcal{H}_n(U) \subset \mathcal{H}_n(\mathbb{C})^m$ belong to the ideal I^m . Therefore $\varphi_U(\tilde{f}) = \varphi_U(\tilde{g})$ by hypothesis. \square

Remark 2.26. This proposition allows to identify \mathcal{H}_n/I with the categorical quotient of \mathcal{H}_n by the ideal I , as defined in the previous subsection.

Corollary 2.27. *The category of analytic algebras in the sense of Malgrange embeds fully faithfully in $\text{AnRing}_{\mathbb{C}}^0$.*

Remark 2.28. This is a low-tech version of the fully faithful embedding of [11, Theorem 12.8].

2.4. Pregeometries of germs. Corollary 2.3 is to some extent a very surprising result, at least for two reasons: it doesn't hold when the topos is different from \mathcal{S} , and it doesn't hold for a general pregeometry, even for $\mathcal{X} = \mathcal{S}$. We will show in this subsection that there is a deeper reason for this result. Namely, we will describe $\text{Str}_{\mathcal{T}_{\text{an}}}^{\text{loc}}(\mathcal{S})$ as algebras for a suitable (multi-sorted) Lawvere theory. We could therefore deduce Corollary 2.3 directly from Lemma 1.14.

Let us define the category $\mathcal{T}_{\text{an}}^0$ as follows:

- (1) the objects of $\mathcal{T}_{\text{an}}^0$ are pairs (n, p) where n is a natural number and $p \in \mathbb{C}^n$.
- (2) A morphism from (n, p) to (m, q) is a germ of holomorphic function from (p, \mathbb{C}^n) to (q, \mathbb{C}^m) .

Observe that the category $\mathcal{T}_{\text{an}}^0$ has products. Indeed, $(n, p) \times (m, q) = (n + m, (p, q))$. Let us say that a morphism $f: (n, p) \rightarrow (m, q)$ in $\mathcal{T}_{\text{an}}^0$ if it can be represented by an open immersion. In particular we have $n = m$. Let $f: (n, p_0) \rightarrow (n, p_1)$ be an admissible morphism and let $g: (m, q) \rightarrow (n, p_1)$ be any other morphism. Then

$$\begin{array}{ccc} (m, q) & \xrightarrow{\text{id}} & (m, q) \\ g + p_0 - p_1 \downarrow & & \downarrow g \\ (n, p_0) & \xrightarrow{f} & (n, p_1) \end{array}$$

is a pullback diagram in $\mathcal{T}_{\text{an}}^0$. In particular, we see that $\mathcal{T}_{\text{an}}^0$ can be equipped with a structure of semi-discrete pregeometry. However, since all admissible morphisms are isomorphisms, we see that $\mathcal{T}_{\text{an}}^0$ is in fact a discrete pregeometry. In other words, $\mathcal{T}_{\text{an}}^0$ is a (multi-sorted) Lawvere theory.

Theorem 2.29. *There exists an equivalence of ∞ -categories $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S}) \simeq \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^0}(\mathcal{S})$.*

Sketch of a proof. Define a functor

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^0}(\mathcal{S}) \rightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})$$

by sending $\mathcal{O}^0 \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^0}(\mathcal{S})$ to the functor $\mathcal{O}: \mathcal{T}_{\mathrm{an}} \rightarrow \mathcal{S}$ defined by

$$\mathcal{O}(U) := \coprod_{p \in U} \mathcal{O}^0(n, p)$$

It is clear that \mathcal{O} is a functor and that it depends functorially on \mathcal{O}^0 . Finally, it commutes with admissible pullbacks and it takes coverings to effective epimorphisms.

On the other direction, if $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S})$ is a $\mathcal{T}_{\mathrm{an}}$ -structure on \mathcal{S} , we define $\mathcal{O}^0: \mathcal{T}_{\mathrm{an}}^0 \rightarrow \mathcal{S}$ as follows:

$$\mathcal{O}(n, p) := \mathcal{O}(\mathbb{C}^n) \times_{\mathbb{C}^n} \{p\}$$

where the map defining this *homotopy* fiber product is $\mathcal{O}(\mathbb{C}^n) \rightarrow \pi_0(\mathcal{O})(\mathbb{C}^n) \xrightarrow{\sigma} \mathbb{C}^n$. It is straightforward to extend this definition on morphisms. Finally, it is clear that these two functors are mutually inverse. \square

2.5. Strict models. It will be of some importance to develop in this section an explicit presentation for the ∞ -category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}^{\mathrm{loc}}(\mathcal{S})$. In virtue of Theorem 2.29, we can replace $\mathcal{T}_{\mathrm{an}}$ with the discrete pregeometry $\mathcal{T}_{\mathrm{an}}^0$. In particular, we have

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}^{\mathrm{loc}}(\mathcal{S}) = \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{an}}^0, \mathcal{S})$$

where the right hand side is the category of functors that preserve *finite* products. Consider now the 1-category $\mathrm{sAnRing} := \mathrm{Funct}^{\times}(\mathcal{T}_{\mathrm{an}}^0, \mathrm{sSet})$ of (strict) functors preserving *finite* products. Invoking [10, 5.5.9.1, 5.5.9.2], we see that defining a morphism $f: F \rightarrow G$ in $\mathrm{sAnRing}$ to be:

- (1) a fibration if it is an objectwise fibration;
- (2) a weak equivalence if it is an objectwise weak equivalence.

we obtain a model structure on $\mathrm{sAnRing}$ whose underlying ∞ -category is precisely $\mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{an}}^0, \mathcal{S})$.

Remark 2.30. Observe that Theorem 2.29 implies in particular that $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathrm{Set}) \simeq \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^0}(\mathrm{Set})$. We therefore deduce an equivalence of 1-categories

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}(\mathrm{sSet}) \simeq \mathrm{Funct}^{\times}(\mathcal{T}_{\mathrm{an}}^0, \mathrm{sSet})$$

Under this equivalence, the object \mathcal{H}_n can be identified with the functor $\mathcal{T}_{\mathrm{an}}^0 \rightarrow \mathrm{sSet}$ corepresented by the germ $(0, \mathbb{C}^n)$. The universal property explained in Proposition 2.23 becomes therefore a direct consequence of the Yoneda lemma.

We will need a better understanding of this model structure.

Lemma 2.31. *Let $f_0: K_0 \rightarrow L_0$ and $f_1: K_1 \rightarrow L_1$ be Kan fibrations (resp. weak equivalences) of simplicial sets. Then $f_0 \times f_1: K_0 \times K_1 \rightarrow L_0 \times L_1$ is a Kan fibration (resp. a weak equivalence).*

Proof. The statement on weak equivalences follows at once from the fact that the geometric realization functor commutes with products and so do the homotopy groups. The statement on Kan fibrations follows immediately from the characterization with the right lifting property against anodyne maps. \square

Proposition 2.32. *Let $f: R \rightarrow S$ be a morphism in $\mathbf{sAnRing}$. The following conditions are equivalent:*

- (1) *the induced morphism $R(0, \mathbb{C}) \rightarrow S(0, \mathbb{C})$ is a fibration (resp. a weak equivalence);*
- (2) *the induced morphism $R(0, \mathbb{C}^n) \rightarrow S(0, \mathbb{C}^n)$ is a fibration (resp. a weak equivalence) for every n ;*
- (3) *the induced morphism $R(p, \mathbb{C}^n) \rightarrow S(p, \mathbb{C}^n)$ is a fibration (resp. a weak equivalence) for every n and every $p \in \mathbb{C}^n$.*

Proof. The implications $3. \Rightarrow 2.$ and $2. \Rightarrow 1.$ are obvious. Suppose now that $R(0, \mathbb{C}) \rightarrow S(0, \mathbb{C})$ is a fibration (resp. a weak equivalence). Since both R and S strictly preserve products, 2. follows from the previous lemma. Finally, 3. follows because we have isomorphisms $(0, \mathbb{C}^n) \simeq (p, \mathbb{C}^n)$. \square

Corollary 2.33. *Every object in $\mathbf{sAnRing}$ is fibrant.*

Proof. Observe that the germ $(0, \mathbb{C})$ has an abelian group structure (given by pointwise sum of germs of holomorphic functions). Since every $F \in \mathbf{sAnRing}$ commutes with products, we see that $F(0, \mathbb{C})$ is a simplicial group and therefore it is a Kan complex. At this point, the conclusion follows directly from Proposition 2.32. \square

Consider the forgetful functor

$$U: \mathbf{sAnRing} \rightarrow \mathbf{sSet}$$

given by $U(R) := R(0, \mathbb{C})$.

Remark 2.34. Observe that this functor is quite different from the underlying algebra functor $\bar{\Phi}$. Indeed, $\bar{\Phi}(R) = \coprod_{p \in \mathbb{C}} R(p, \mathbb{C})$.

U commutes with limits and sifted colimits. In particular, since both $\mathbf{sAnRing}$ and \mathbf{sSet} are presentable, it follows that it has a left adjoint, which we denote by

$$\mathcal{H}\{-\}: \mathbf{sSet} \rightarrow \mathbf{sAnRing}$$

It follows from Proposition 2.32 that U is a right Quillen functor, so that $\mathcal{H}\{-\} \dashv U$ is a Quillen adjunction.

Proposition 2.35. *The collection of morphisms $I := \{\mathcal{H}\{\partial\Delta^n\} \rightarrow \mathcal{H}\{\Delta^n\}\}_{n \in \mathbb{N}}$ is a set of generating cofibrations for $\mathbf{sAnRing}$. The family of morphisms $J := \{\mathcal{H}\{\Lambda_i^n\} \rightarrow \mathcal{H}\{\Delta^n\}\}_{n \in \mathbb{N}, 0 \leq i \leq n}$ is a set of generating trivial cofibrations for $\mathbf{sAnRing}$.*

Proof. We already remarked that $\mathcal{H}\{-\}$ is a left Quillen functor. It follows that all the morphisms in J are trivial cofibrations and all the morphisms in I are cofibrations. Let now $p: R \rightarrow S$ a morphism in $\mathbf{sAnRing}$. Observe that the lifting problems

$$\begin{array}{ccc} \mathcal{H}\{\Lambda_i^n\} & \longrightarrow & R \\ \downarrow & \nearrow & \downarrow p \\ \mathcal{H}\{\Delta^n\} & \longrightarrow & S \end{array} \quad \begin{array}{ccc} \mathcal{H}\{\partial\Delta^n\} & \longrightarrow & R \\ \downarrow & \nearrow & \downarrow p \\ \mathcal{H}\{\Delta^n\} & \longrightarrow & S \end{array}$$

are respectively equivalent to the lifting problems

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & U(R) \\ \downarrow & \nearrow & \downarrow U(f) \\ \Delta^n & \longrightarrow & U(S) \end{array} \quad \begin{array}{ccc} \partial\Delta^n & \longrightarrow & U(R) \\ \downarrow & \nearrow & \downarrow U(f) \\ \Delta^n & \longrightarrow & U(S) \end{array}$$

In particular, we see that p has the right lifting property with respect to maps in J (resp. I) if and only if $U(p)$ is a fibration (resp. a trivial fibration). This completes the proof. \square

Lemma 2.36. *Let I_n be the set with n elements. Then $\pi_0(\mathcal{H}\{I_n\}) = \mathcal{H}_n$.*

Proof. The two universal properties match. \square

Proposition 2.37. *For every n , we have*

$$\pi_i(\overline{\Phi}(\mathcal{H}\{\Delta^n\})) = \begin{cases} \mathbb{C}\{z\} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathbb{C}\{z\}$ denotes the algebra of germs of holomorphic functions around $0 \in \mathbb{C}$.

Proof. The morphism $\Delta^0 \rightarrow \partial\Delta^n$ selecting the 0-th vertex is an acyclic cofibration. It follows that $\overline{\Phi}(\mathcal{H}\{\partial\Delta^n\})$ is weakly equivalent to $\overline{\Phi}(\mathcal{H}\{\Delta^0\})$. It is therefore sufficient to prove the proposition for $n = 0$. The case $i = 0$ follows directly from the previous lemma. If $i \geq 1$, we first observe that it is sufficient to show that $\pi_i(\mathcal{H}\{\Delta^0\}(0, \mathbb{C})) = 0$. For a simplicial set K with finitely many simplexes in every degree, the n -simplexes of $\mathcal{H}\{\Delta^0\}(0, \mathbb{C})$ are in bijection with $\mathbb{C}\{z_1, \dots, z_m\}_0$, where

the variables z_j correspond to the n -simplexes of K . In particular, we see that $\mathcal{H}\{\Delta^0\}$ is the constant simplicial set associated to $\mathbb{C}\{z\}_0$. The proof is therefore complete. \square

Proposition 2.38. *We have $\pi_0(\overline{\Phi}(\mathcal{H}\{\partial\Delta^n\})) = \mathbb{C}\{z\}$.*

Proof. It will be enough to prove that $\pi_0(\mathcal{H}\{\partial\Delta^n\}(0, \mathbb{C}))$ can be identified with the set of germs of holomorphic maps $(0, \mathbb{C}) \rightarrow (0, \mathbb{C})$. We can explicitly represent $\mathcal{H}\{\partial\Delta^n\}(0, \mathbb{C})$ as a simplicial algebra whose 1-skeleton is

$$\mathbb{C}\{z_{0,0}, z_{0,1}, \dots, z_{i,j}, \dots, z_{n-1,n}\}_0 \rightarrow \mathbb{C}\{z_0, \dots, z_n\}_0$$

where the variable z_{ij} corresponds to the edge of $\partial\Delta^n$ connecting the vertex i to the vertex j (hence $i \leq j$) and

$$d_0(z_{i,j}) = z_i, \quad d_1(z_{i,j}) = -z_j, \quad s_0(z_i) = z_{i,i}$$

Applying Dold-Kan and computing the 0-th cohomology of the resulting complex, we deduce that

$$\pi_0(\mathcal{H}\{\partial\Delta^n\}(0, \mathbb{C})) = \mathbb{C}\{z_0, \dots, z_n\}_0 / (z_i + z_j)_{0 \leq i < j \leq n} \simeq \mathbb{C}\{z\}_0$$

\square

3. THE FUNCTOR OF POINTS

With Definition 1.3 we are giving a “structured space” perspective on derived \mathbb{C} -analytic spaces. However, in the practice, it is often important to rely on the dual perspective of the functor of points. The goal of this section is precisely to introduce this alternative point of view and to prove a comparison result with Definition 1.3. Concretely, this means that we will have to discuss the following points:

- (1) we have to exhibit a geometric context $(\mathcal{C}, \tau, \mathbf{P})$ (in the precise sense of [20, Definition 2.11]) in such a way that $\mathrm{dAn}_{\mathcal{C}}$ can be exhibited as a full subcategory of $\mathrm{Sh}(\mathcal{C}, \tau)$.
- (2) We will have to identify the essential image of (a large subcategory of) $\mathrm{dAn}_{\mathcal{C}}$ with the geometric stacks in the sense of [20, Definition 2.15].

In dealing with the first point of the list, one could try to invoke [12, Theorem 2.4.1]. Unfortunately, this result is not sufficient for our purposes: indeed, there is discrepancy between the $\mathcal{T}_{\mathrm{an}}$ -schemes and the derived \mathbb{C} -analytic spaces (see [11, Corollary 12.22 and Proposition 12.23] for a detailed discussion of this difference). Nevertheless, the same idea of the proof in loc. cit. applies to our context, as we are going to discuss.

3.1. The geometric context.

Definition 3.1. We let $\text{Stn}_{\mathbb{C}}^{\text{der}}$ be the full subcategory of $\text{dAn}_{\mathbb{C}}$ spanned by those derived \mathbb{C} -analytic spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}^{\text{alg}})$ is of the form $\Phi(S)$, with S a Stein space. We will refer to $\text{Stn}_{\mathbb{C}}^{\text{der}}$ as the category of derived (\mathbb{C} -analytic) Stein spaces.

The notion of étale morphism of \mathcal{T}_{an} -structured topoi (see Definition 1.6) induces a Grothendieck topology τ on the ∞ -category $\text{Stn}_{\mathbb{C}}^{\text{der}}$. We define a functor $\tilde{\phi}: \text{dAn}_{\mathbb{C}} \rightarrow \text{PSh}(\text{Stn}_{\mathbb{C}}^{\text{der}})$ as follows:

$$\text{dAn}_{\mathbb{C}} \xrightarrow{j} \text{Fun}(\text{dAn}_{\mathbb{C}}^{\text{op}}, \mathcal{S}) \rightarrow \text{Fun}((\text{Stn}_{\mathbb{C}}^{\text{der}})^{\text{op}}, \mathcal{S})$$

Our first task is to show that $\tilde{\phi}$ is fully faithful. To achieve this, we will show that it can be factored as

$$\phi: \text{dAn}_{\mathbb{C}} \rightarrow \text{Sh}(\text{Stn}_{\mathbb{C}}^{\text{der}}, \tau)$$

and that moreover ϕ is a fully faithful functor. We will need a couple of preliminary facts.

Lemma 3.2. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a derived \mathbb{C} -analytic space. Then \mathcal{X} is hypercomplete.*

Proof. Let \mathcal{Y} be an ∞ -topos. The hypercompletion of \mathcal{Y} is defined to be the full subcategory \mathcal{Y}^{\wedge} of \mathcal{Y} spanned by hypercomplete objects. Moreover, \mathcal{Y} is said to be hypercomplete if the inclusion $\mathcal{Y}^{\wedge} \subset \mathcal{Y}$ is a categorical equivalence. Therefore, it follows from [10, 6.5.2.21–6.5.2.22] that being hypercomplete is local on \mathcal{Y} , in the sense that if there are objects $U_i \in \mathcal{Y}$ such that the map $\coprod U_i \rightarrow \mathbf{1}_{\mathcal{Y}}$ is an effective epimorphism and each ∞ -topos $\mathcal{Y}_{/U_i}$ is hypercomplete, then the same goes for \mathcal{Y} .

In the case of our interest, the definition of derived \mathbb{C} -analytic space allows us to choose objects $\coprod U_i \in \mathcal{X}$ such that each $(\mathcal{X}_{/U_i}, \pi_0 \mathcal{O}_{\mathcal{X}}^{\text{alg}}|_{U_i})$ is a local model for \mathbb{C} -analytic spaces. In particular, we can identify $\mathcal{X}_{/U_i} \simeq \text{Sh}(X_i)$, where X_i is a locally compact Hausdorff space of finite covering dimension. It follows that the homotopy dimension of X_i is finite and therefore that $\mathcal{X}_{/U_i}$ is hypercomplete. \square

Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a derived \mathbb{C} -analytic space and suppose that \mathcal{X} is 0-localic. We will denote by X^{top} the underlying topological space of \mathcal{X} (see [12, 2.5.21]). As every point in a \mathbb{C} -analytic space has a Stein open neighbourhood, we can find an hypercover U^{\bullet} of X^{top} made by open Stein subsets of X^{top} . Each open subset of X^{top} defines a (discrete) object of the ∞ -topos $\mathcal{X} = \text{Sh}(X^{\text{top}})$. Therefore, the simplicial object U^{\bullet} determines by composition with the Yoneda embedding a simplicial object in X^{top} , which we will still denote U^{\bullet} . It follows directly from the definitions and from the criterion [10, 7.2.1.14] that U^{\bullet} is an hypercover of \mathcal{X} . Let

us denote by X^n the derived \mathbb{C} -analytic space defined as

$$X^n := (\mathcal{X}/_{U^n}, \mathcal{O}_X|_{U^n})$$

The universal property of étale morphisms of structured \mathcal{T}_{an} -topoi (see [12, Remark 2.3.4]) shows that we can arrange the X^n into a simplicial object X^\bullet in the ∞ -category $\text{Top}(\mathcal{T}_{\text{an}})$. We claim that the geometric realization of the diagram X^\bullet in $\text{Top}(\mathcal{T}_{\text{an}})$ (and hence in $\text{dAn}_{\mathbb{C}}$) coincides precisely with the original derived \mathbb{C} -analytic space X . To prove this it is sufficient to show the following two assertions:

- (1) one has $|\mathcal{X}/_{U^\bullet}| \simeq \mathcal{X}$;
- (2) Let $j_*^n: \mathcal{X}/_{U^\bullet} \rightarrow \mathcal{X}$ be the given geometric morphism. Then in $\text{Str}_{\mathcal{T}_{\text{an}}}^{\text{loc}}(\mathcal{X})$ one has

$$\mathcal{O}_X \simeq \varprojlim_{\Delta} j_*^n \mathcal{O}_X|_{U^n}$$

The first assertion is a consequence of the general descent theory for ∞ -topoi (see [10, Theorem 6.1.3.9]) and the fact that in \mathcal{X} one has

$$|U^\bullet| \simeq U$$

as it follows from Lemma 3.2 and from [10, 6.5.3.12]. As for the second statement, since $\text{Str}_{\mathcal{T}_{\text{an}}}^{\text{loc}}(\mathcal{X})$ is closed under limits in $\text{Fun}(\mathcal{T}_{\text{an}}, \mathcal{X})$, it will be enough to show that for every $V \in \mathcal{T}_{\text{an}}$, in \mathcal{X} one has

$$\mathcal{O}_X(V) \simeq \varprojlim_{\Delta} j_*^n \mathcal{O}_X(V)|_{U^n}$$

Since $\mathcal{X} = \text{Sh}(X^{\text{top}})$ is closed under limits in $\text{PSh}(X^{\text{top}})$, we see that it is enough to prove that whenever W is an open of X^{top} one has

$$\mathcal{O}_X(V)(W) \simeq \varprojlim_{\Delta} \mathcal{O}_X(V)(U^n \times W)$$

Since $U^\bullet \times W$ is an hypercover of W , this statement is equivalent to say that each $\mathcal{O}_X(V) \in \mathcal{X}$ is hypercomplete, which is obvious since \mathcal{X} is itself hypercomplete in virtue of Lemma 3.2.

Summarizing, we proved the following:

Lemma 3.3. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived \mathbb{C} -analytic space such that \mathcal{X} is 0-localic. Then there exists an hypercover U^\bullet in \mathcal{X} such that each derived \mathbb{C} -analytic space $X^n := (\mathcal{X}/_{U^n}, \mathcal{O}_X|_{U^n})$ is a derived Stein space. Moreover, for each such hypercover, the geometric realization of X^\bullet in $\text{dAn}_{\mathbb{C}}$ is X .*

Corollary 3.4. *The Grothendieck topology τ on $\text{Stn}_{\mathbb{C}}^{\text{der}}$ is subcanonical. Moreover, if $X = (\mathcal{X}, \mathcal{O}_X)$ is a derived \mathbb{C} -analytic space, then $\tilde{\phi}(X)$ belongs to the hypercompletion of $\text{Sh}(\text{Stn}_{\mathbb{C}}^{\text{der}}, \tau)$.*

One can prove something more general. Indeed, the notion of étale morphism of \mathcal{T}_{an} -structures defines also a Grothendieck topology τ' on $\text{dAn}_{\mathbb{C}}$ and, with the same arguments used above, one can prove that the Yoneda embedding

$$j: \text{dAn}_{\mathbb{C}} \rightarrow \text{PSh}(\text{dAn}_{\mathbb{C}})$$

factors through $\text{Sh}(\text{dAn}_{\mathbb{C}}, \tau')$. This has the following useful consequence:

Corollary 3.5. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived \mathbb{C} -analytic space and let $p: U \rightarrow \mathbf{1}_X$ be an effective epimorphism. Let U^\bullet be the Čech nerve of p and set $X^n := (\mathcal{X}_{/U^n}, \mathcal{O}_X|_{U^n})$. Then in $\text{Sh}(\text{Stn}_{\mathbb{C}}^{\text{der}}, \tau)$ one has*

$$\phi(X) \simeq \text{colim}_{\Delta} \phi(X^\bullet)$$

Proof. Let us temporarily denote by $\psi: \text{dAn}_{\mathbb{C}} \rightarrow \text{Sh}(\text{dAn}_{\mathbb{C}}, \tau')$ the functor obtained by factorizing the Yoneda embedding $j: \text{dAn}_{\mathbb{C}} \rightarrow \text{PSh}(\text{dAn}_{\mathbb{C}})$. The above discussion makes clear that the relation

$$\psi(X) \simeq \text{colim}_{\Delta} \psi(X^\bullet)$$

holds in $\text{Sh}(\text{dAn}_{\mathbb{C}}, \tau')$. The morphism of sites $(\text{Stn}_{\mathbb{C}}^{\text{der}}, \tau) \rightarrow (\text{dAn}_{\mathbb{C}}, \tau')$ is both continuous and cocontinuous. It follows from [20, Lemma 2.30] that the restriction along this functor is a left adjoint. In particular, it commutes with colimits, so that the proof is complete. \square

Proposition 3.6. *The functor ϕ is fully faithful.*

Proof. Let $X, Y \in \text{dAn}_{\mathbb{C}}$ and consider the natural map

$$\psi_{X,Y}: \text{Map}_{\text{dAn}_{\mathbb{C}}}(X, Y) \rightarrow \text{Map}_{\text{Sh}(\text{Stn}_{\mathbb{C}}^{\text{der}}, \tau)}(\phi(X), \phi(Y))$$

Keeping Y fixed, consider the full subcategory \mathcal{C} of $\text{dAn}_{\mathbb{C}}$ spanned by those X for which $\psi_{X,Y}$ is an equivalence.

Choose objects U_i of $X = (\mathcal{X}, \mathcal{O}_X)$ in such a way that $p: \coprod U_i \rightarrow \mathbf{1}_X$ is an effective epimorphism and that each $\mathcal{X}_{/U_i}$ is 0-localic. Let $U := \coprod U_i$ and let U^\bullet be the Čech nerve of p . Finally, set $X^n := (\mathcal{X}_{/U^n}, \mathcal{O}_X|_{U^n})$. It follows from Corollary 3.5 that

$$\phi(X) \simeq \text{colim}_{\Delta} \phi(X^n)$$

We can therefore reduce ourselves to the case where \mathcal{X} is 0-localic. Invoking Lemma 3.3 and repeating the above argument, we can further reduce to the case where $X \in \text{Stn}_{\mathbb{C}}^{\text{der}}$, and the lemma is now a restatement of the Yoneda lemma. \square

3.2. Geometric stacks. We now turn to the second main goal of this section, that is, the characterization of the essential image of $\phi: \mathrm{dAn}_{\mathbb{C}} \rightarrow \mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau)$. First of all, let us observe that letting $\mathbf{P}_{\mathrm{\acute{e}t}}$ be the collection of étale morphisms in $\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}$, the triple $(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau, \mathbf{P}_{\mathrm{\acute{e}t}})$ becomes a geometric context in the sense of [20, Definition 2.11]. We will refer to geometric stacks relative to this context as *derived Deligne-Mumford analytic stacks*. We will denote by DM the full subcategory of $\mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau)^{\wedge}$ spanned by derived Deligne-Mumford analytic stacks. We will further denote by DM_n the full subcategory of DM spanned by n -geometric stacks. On the other side, we will denote by $\mathrm{dAn}_{\mathbb{C}}^{\leq n}$ the full subcategory of $\mathrm{dAn}_{\mathbb{C}}$ spanned by those derived \mathbb{C} -analytic spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ for which \mathcal{X} is an n -localic topos. Finally, we will denote by $\mathrm{dAn}_{\mathbb{C}}^{\mathrm{loc}}$ the reunion of all the subcategories $\mathrm{dAn}_{\mathbb{C}}^{\leq n}$ as n ranges through the integers. With these notations, we can formulate the main result of this section:

Theorem 3.7. *For every $n \geq 0$, the functor $\phi: \mathrm{dAn}_{\mathbb{C}} \rightarrow \mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau)$ restricts to an equivalence $\mathrm{dAn}_{\mathbb{C}}^{\leq n} \simeq \mathrm{DM}_n$.*

Remark 3.8. The previous theorem (when combined with Corollary 3.10) provides a large generalization of the comparison result [11, Theorem 12.8]. See however Remark 12.25 loc. cit. for something going in this direction.

The following lemma will be used repeatedly in the proof of Theorem 3.7.

Lemma 3.9. *Let $f: U \rightarrow V$ be a morphism in $\mathcal{T}_{\mathrm{an}}$. Let us write $\mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(U) = (\mathcal{X}_U, \mathcal{O}_U)$ and $\mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V) = (\mathcal{X}_V, \mathcal{O}_V)$. The following conditions are equivalent:*

- (1) *f is étale (resp. an étale monomorphism);*
- (2) *the induced morphism $(f_*, \varphi): \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(U) \rightarrow \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V)$ is an étale morphism of derived \mathbb{C} -analytic spaces (resp. is étale and $\mathcal{X}_U \simeq (\mathcal{X}_V)_{/W}$ where W is a discrete object of \mathcal{X}_V).*

Proof. Suppose first that f is étale. Then the assertion follows immediately from [12, Example 2.3.8]. If moreover f is a monomorphism, then \mathcal{X}_U can be identified with the topos associated to an open subset W of V^{top} and therefore $\mathcal{X}_U \simeq (\mathcal{X}_V)_{/W}$ and W is clearly a discrete object of \mathcal{X}_V . Vice-versa, suppose that $(f_*, \varphi): \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(U) \rightarrow \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V)$ is an étale morphism of derived \mathbb{C} -analytic spaces. Then f can be recovered entirely from the geometric morphism $f_*: \mathcal{X}_U \rightarrow \mathcal{X}_V$ and the very definition of étale morphisms of ∞ -topoi makes clear that f has to be a local homeomorphism, which is injective if \mathcal{X}_U is the étale subtopos of \mathcal{X}_V associated to a discrete object. Since it was a holomorphic map to begin with, it also follows that f is a local biholomorphism, thus completing the proof. \square

The proof of Theorem 3.7 naturally splits into two parts:

- (1) to show that whenever $X \in \mathrm{dAn}_{\mathbb{C}}^{\leq n+1}$ the sheaf $\phi(X)$ is an n -geometric stack;
- (2) to show that every n -geometric stack arises in this way.

The rest of this subsection will be devoted to the proof of the first step. We will deal with the second one in Section 3.4, after having discussed the notion of truncation at length in Section 3.3.

Corollary 3.4 shows that ϕ factors in fact through $\mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau)^{\wedge}$, and therefore we only have to prove that $\phi(X)$ admits an n -atlas and that the diagonal of $\phi(X)$ is $(n-1)$ -representable. In order to simplify the proof, it will be convenient for the rest of this section to introduce the category $\mathrm{dAn}_{\mathbb{C}}^0$ of 0-localic derived \mathbb{C} -analytic spaces. We will endow $\mathrm{dAn}_{\mathbb{C}}^0$ with the étale topology σ and we will let $\mathbf{Q}_{\mathrm{\acute{e}t}}$ be the collection of étale morphisms. The inclusion

$$(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau, \mathbf{P}_{\mathrm{\acute{e}t}}) \rightarrow (\mathrm{dAn}_{\mathbb{C}}^0, \sigma, \mathbf{Q}_{\mathrm{\acute{e}t}})$$

is a morphism of geometric contexts, and every object in $\mathrm{dAn}_{\mathbb{C}}^0$ defines a 1-geometric stack for the Stein context. Lemma 3.3 shows that every object in $\mathrm{dAn}_{\mathbb{C}}^0$ admits an atlas coming from $\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}$, and the morphism of sites $(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau) \rightarrow (\mathrm{dAn}_{\mathbb{C}}^0, \sigma)$ is easily seen to be cocontinuous. Therefore we can apply [20, Lemma 2.36] and temporarily work with the geometric stacks for the second context. This has an important advantage that we are going to describe. The geometric context $(\mathrm{dAn}_{\mathbb{C}}^0, \sigma, \mathbf{Q}_{\mathrm{\acute{e}t}})$ is closed under σ -descent in the sense that whenever we are given a morphism $F \rightarrow G$ in $\mathrm{Sh}(\mathrm{dAn}_{\mathbb{C}}^0, \tau)^{\wedge}$ with G being representable, if there exists a τ -covering $G_i \rightarrow G$ (with G_i being representable) such that each base change $G_i \times_G F$ is representable, then the same goes for F . This is easy to see: indeed, we have $G_i = \phi(U_i)$ for certain $U_i \in \mathrm{dAn}_{\mathbb{C}}^0$ and $G = \phi(Y)$. Since ϕ is fully faithful in virtue of Proposition 3.6, the morphisms $G_i \rightarrow G$ are represented by morphisms $U_i \rightarrow Y$ which are étale by Lemma 3.9. Let $U := \coprod U_i$, $p: U \rightarrow Y$ the total morphism and U^{\bullet} the Čech nerve of p . Since ϕ commutes with limits and with disjoint unions, we see that $\phi(U^{\bullet})$ is the Čech nerve of $\coprod G_i \rightarrow G$. By hypothesis, each level of the simplicial object $\phi(U^{\bullet}) \times_G F$ is representable. Since ϕ is fully faithful, we can form a simplicial object V^{\bullet} in $\mathrm{dAn}_{\mathbb{C}}^0$ in such a way that $\phi(V^{\bullet}) \simeq \phi(U^{\bullet}) \times_G F$. Lemma 3.9 shows that all the face maps in V^{\bullet} are étale. As consequence, the geometric realization of V^{\bullet} exists in $\mathrm{dAn}_{\mathbb{C}}^0$. Let us denote by X this colimit. Corollary 3.5 shows that

$$\phi(X) \simeq |\phi(V^{\bullet})| \simeq |\phi(U^{\bullet}) \times_G F| \simeq F.$$

Thus the proof of the claim is completed.

The importance of this fact is the following: since the geometric context $(\mathrm{dAn}_{\mathbb{C}}^0, \sigma, \mathbf{Q}_{\mathrm{\acute{e}t}})$ is closed under σ -descent, the tool of groupoid presentations becomes available for geometric stacks on this context. In particular, the requirement on

the representability of the diagonal in the definition of geometric stack is now superfluous (cf. [26, Remark 1.3.3.2]). Therefore, in proving that $\phi(X)$ is geometric with respect to this context, we will only need to show that it admits an n -atlas. Taking into account the shift of the geometric level coming from [20, Lemma 2.36], we have now to show that whenever $n \geq -1$ and $X \in \mathrm{dAn}_{\mathbb{C}}^{\leq n+1}$, then $\phi(X)$ is n -geometric.

Proof of Theorem 3.7, Step 1. Let X be an $(n+1)$ -localic derived \mathbb{C} -analytic space. We will prove the statement by induction on n . If $n = -1$, then $\phi(X)$ coincides with the representable sheaf associated to X itself. Therefore, $\phi(X)$ is (-1) -geometric and hence the base of the induction holds.

Let now $n \geq 0$ and suppose that the statement has already been proved for n -localic derived \mathbb{C} -analytic spaces. Fix $X = (\mathcal{X}, \mathcal{O}_X) \in \mathrm{dAn}_{\mathbb{C}}^{\leq n+1}$. Choose an effective epimorphism $\coprod U_i \rightarrow \mathbf{1}_X$ in such a way that ∞ -topos $\mathcal{X}_{/U_i}$ is 0-localic. We claim that each U_i is n -truncated. Assuming for the moment this fact, we see that the morphisms $\phi(U_i) \rightarrow \phi(X)$ are étale and the total morphism $\coprod \phi(U_i) \rightarrow \phi(X)$ is an effective epimorphism in $\mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau)^\wedge$. Since each U_i is n -truncated, we conclude that the morphisms $\phi(U_i) \rightarrow \phi(X)$ are representable by sheaves of the form $\phi(Y)$ with $Y \in \mathrm{dAn}_{\mathbb{C}}^{\leq n}$. The inductive hypothesis implies that each $\phi(Y)$ is $(n-1)$ -geometric and therefore we obtain that $\phi(X)$ is n -geometric.

We are therefore left to prove the claim. To do so, we replace X with $t_0(X) := (\mathcal{X}, \pi_0 \mathcal{O}_X)$. We can review the latter as a $\mathcal{G}_{\mathrm{an}}^0$ -structured topos, where $\mathcal{G}_{\mathrm{an}}^0$ is a 0-truncated geometric envelope for the pregeometry $\mathcal{T}_{\mathrm{an}}$. [12, Lemma 2.6.19] shows that $(\mathcal{X}, \pi_0 \mathcal{O}_X)$ is an $(n+1)$ -truncated object of $\mathrm{Top}(\mathcal{G}_{\mathrm{an}}^0)$. Let F be the functor on $\mathrm{Stn}_{\mathbb{C}}$ represented by $t_0(X)$ and let F_i the one represented by $t_0(U_i)$. It is enough to show that the fibers of $F_i(S) \rightarrow F(S)$ are n -truncated for every Stein space S . Since $t_0(X)$ is $(n+1)$ -truncated, we see that $F(S)$ is $(n+1)$ -truncated. On the other side, $F_i(S)$ is 0-truncated. The assertion now follows from the long exact sequence of homotopy groups. \square

Before discussing the essential surjectivity, we will need a digression on the truncation functor for derived \mathbb{C} -analytic spaces.

3.3. Truncations of derived \mathbb{C} -analytic spaces. Let $(\mathrm{Stn}, \tau_0, \mathbf{P}_0)$ be the geometric context introduced in [20, §3.2]. Observe that there is a continuous morphism of geometric contexts $u: (\mathrm{Stn}, \tau_0, \mathbf{P}_0) \rightarrow (\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau, \mathbf{P}_{\mathrm{ét}})$. This functor is fully faithful in virtue of [11, Theorem 12.8]. It follows that there exists a fully faithful functor

$$u_s: \mathrm{Sh}(\mathrm{Stn}, \tau_{\mathrm{qét}}) \rightarrow \mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau_{\mathrm{qét}})$$

which moreover preserves geometric stacks (we refer to [20, §2.4] for a discussion of the notation employed). Conversely, using Theorem 3.7 we have:

Corollary 3.10. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be an n -localic 0-truncated derived \mathbb{C} -analytic space. Then $\phi(X) = (\mathcal{X}, \mathcal{O}_X) \in \mathrm{Sh}(\mathrm{Stn}_{\mathbb{C}}^{\mathrm{der}}, \tau_0)$ belongs to the essential image of u_s and is a higher analytic Deligne-Mumford stack in the sense of [20].*

Proof. We already know that $\phi(X)$ is geometric. To prove that $\phi(X)$ belongs to the essential image of u_s , we can proceed by induction on n . When $n = 0$, the statement is clear, and the induction step follows from the construction of the atlas of $\phi(X)$ given in the proof of Theorem 3.7 and the fact that u_s commutes with geometric realizations of étale groupoids (being a left adjoint). \square

One of the most basic and yet useful constructions in derived geometry is the truncation of a derived object. This often allows to reduce the proofs to the classical setting, where they can be handled with different techniques. As we will see, this is exactly the case for many of the main results of this article. For this reason, we introduce now the truncation functor t_0 . Roughly speaking this has simply to be the functor sending a derived \mathbb{C} -analytic space $(\mathcal{X}, \mathcal{O}_X)$ into $(\mathcal{X}, \tau_{\leq 0}\mathcal{O}_X)$ (that the latter is still a derived \mathbb{C} -analytic space is a consequence of [12, Proposition 3.3.3] and of [11, Proposition 11.4]). However, in order to construct this as an ∞ -functor we will need to describe it in a rather different fashion.

As it is consequence of a much more general fact concerning geometries, let us switch for a short while to this setting. Let \mathcal{G} be a geometry (e.g. any geometric envelope for $\mathcal{T}_{\mathrm{an}}$) and let \mathcal{X} be an ∞ -topos. Inside $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ we can look for the full subcategory spanned by n -truncated objects. Let us denote it by $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})_{\leq n}$. A very natural question is whether the latter category can be obtained as the category of structures for a suitable modification of \mathcal{G} . This is indeed the case; the relevant object is referred to as the n -stub of \mathcal{G} and its existence is guaranteed by [12, Proposition 1.5.11]. Let us denote it by $\mathcal{G}_{\leq n}$. By construction, it comes equipped with a morphism of geometries $\mathcal{G} \rightarrow \mathcal{G}_{\leq n}$. When $n = 0$ we can ask whether the relative spectrum functor associated to such a morphism coincides with the functor t_0 we are trying to define. This is probably not true without additional hypotheses on \mathcal{G} . The point is that, in general, it is not true that if \mathcal{O} is a \mathcal{G} -structure on an ∞ -topos \mathcal{X} then $\tau_{\leq n} \circ \mathcal{O}$ is again a \mathcal{G} -structure. There is a sufficient condition for this to be true, though: it happens when the geometry is compatible with n -truncations (see [12, Definition 3.3.2]). Under this condition we are able to prove:

Proposition 3.11. *Let \mathcal{G} be a geometry compatible with n -truncations and let $\mathcal{G} \rightarrow \mathcal{G}_{\leq n}$ be an n -stub for \mathcal{G} . Then*

$$\mathrm{Spec}_{\mathcal{G}}^{\mathcal{G}_{\leq n}} : \mathcal{T}\mathrm{op}(\mathcal{G}) \rightarrow \mathcal{T}\mathrm{op}(\mathcal{G}_{\leq n})$$

coincides on objects with the assignment

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$$

Proof. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a \mathcal{G} -structured topos. It follows from [12, Proposition 3.3.3] that $\tau_{\leq n} \mathcal{O}_{\mathcal{X}}$ is a \mathcal{G} -structure on \mathcal{X} . Since $\tau_{\leq n} \mathcal{O}_{\mathcal{X}}$ is n -truncated, [12, Proposition 1.5.14] shows that it defines a $\mathcal{G}_{\leq n}$ -structure on \mathcal{X} . The morphism $\mathcal{O}_{\mathcal{X}} \rightarrow \tau_{\leq n} \mathcal{O}_{\mathcal{X}}$ is a local morphism because \mathcal{G} is compatible with n -truncations. Therefore, it defines a well defined morphism

$$p_n : (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \rightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

in $\mathcal{T}\mathrm{op}(\mathcal{G})$. We claim that for every $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in \mathcal{T}\mathrm{op}(\mathcal{G}_{\leq n})$, the canonical morphism

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}(\mathcal{G}_{\leq n})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})) \rightarrow \mathrm{Map}_{\mathcal{T}\mathrm{op}(\mathcal{G})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$$

is a homotopy equivalence. Indeed, we have a commutative diagram of fiber sequences

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Str}_{\mathcal{G}_{\leq n}}^{\mathrm{loc}}(\mathcal{Y})}(f^{-1} \tau_{\leq n} \mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{Y}}) & \longrightarrow & \mathrm{Map}_{\mathrm{Str}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{Y})}(f^{-1} \mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{Y}}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{T}\mathrm{op}(\mathcal{G}_{\leq n})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})) & \longrightarrow & \mathrm{Map}_{\mathcal{T}\mathrm{op}(\mathcal{G})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{T}\mathrm{op}}(\mathcal{Y}, \mathcal{X}) & \xrightarrow{\mathrm{id}} & \mathrm{Map}_{\mathcal{T}\mathrm{op}}(\mathcal{Y}, \mathcal{X}) \end{array}$$

where both the fibers are computed over the geometric morphism $f^{-1} : \mathcal{X} \rightleftarrows \mathcal{Y} : f_*$. Since f^{-1} is left exact, we see that

$$f^{-1} \tau_{\leq n} \mathcal{O}_{\mathcal{X}} \simeq \tau_{\leq n} f^{-1} \mathcal{O}_{\mathcal{X}}$$

Finally, since the functor $\mathrm{Str}_{\mathcal{G}_{\leq n}}^{\mathrm{loc}}(\mathcal{Y}) \rightarrow \mathrm{Str}_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{Y})$ is fully faithful (in virtue of [12, Proposition 1.5.14]), we see that the top horizontal morphism is a homotopy equivalence. The proof is therefore complete. \square

Definition 3.12. Let \mathcal{G} be a geometry compatible with 0-truncations. We will refer to the functor $\mathrm{Spec}_{\mathcal{G}}^{\mathcal{G}_{\leq 0}}$ as the *truncation functor* and we will denote it $t_0^{\mathcal{G}}$, or simply by t_0 when the geometry \mathcal{G} is clear from the context.

The analytic pregeometry \mathcal{T}_{an} is compatible with n -truncations for every $n \geq 0$ in virtue of [11, Proposition 11.4]. Therefore this allows to introduce the truncation functor for derived \mathbb{C} -analytic spaces. Let us denote by $\text{dAn}_{\mathbb{C}}^0$ the full subcategory of $\text{dAn}_{\mathbb{C}}$ spanned by the 0-truncated derived \mathbb{C} -analytic spaces. Similarly, let us denote by $\text{Top}^0(\mathcal{T}_{\text{an}})$ the full subcategory of $\text{Top}(\mathcal{T}_{\text{an}})$ spanned by 0-truncated \mathcal{T}_{an} -structured topoi. Then we have (cf. [26, Proposition 2.2.4.4]):

Proposition 3.13. *Let $i: \text{dAn}_{\mathbb{C}}^0 \rightarrow \text{dAn}_{\mathbb{C}}$ be the natural inclusion functor. Then:*

- (1) *the functor $t_0: \text{Top}(\mathcal{T}_{\text{an}}) \rightarrow \text{Top}^0(\mathcal{T}_{\text{an}})$ restricts to a functor $t_0: \text{dAn}_{\mathbb{C}} \rightarrow \text{dAn}_{\mathbb{C}}^0$;*
- (2) *the functor i is left adjoint to the functor t_0 ;*
- (3) *the functor i is fully faithful.*

Proof. It follows from inspection that t_0 respects the category of derived \mathbb{C} -analytic spaces. Therefore the points (1) and (2) follow immediately. As for (3), the result follows from the description of the unit of this adjunction given in Proposition 3.11 and the fact that the truncation functor $\tau_{\leq 0}$ of any ∞ -topos is idempotent. \square

We close this section with a couple of important remark. Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an n -localic 0-truncated derived \mathbb{C} -analytic space. If $Y \rightarrow X$ is an étale morphism of derived \mathbb{C} -analytic spaces, we see that Y has to be 0-truncated. This observation together with the fully faithfulness of the functor ϕ proved in Proposition 3.6 yields the following result (cf. [26, Proposition 2.2.4.4.(4)]):

Corollary 3.14. *Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an n -localic 0-truncated derived \mathbb{C} -analytic space. Then the small étale site $(\text{Stn}_{\mathbb{C}/X}^{\text{der}})_{\text{ét}}$ is canonically equivalent to the small étale site $(\text{Stn}_{\mathbb{C}/\phi(X)})_{\text{ét}}$.*

We also have the following useful equivalence, very familiar to the panorama of derived algebraic geometry (cf. [26, Corollary 2.2.2.9]):

Proposition 3.15. *Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a derived \mathbb{C} -analytic space and let $t_0(X) = (\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ be its truncation. The base change along the morphism $t_0(X) \rightarrow X$ induces an equivalence*

$$(\text{Stn}_{\mathbb{C}/X}^{\text{der}})_{\text{ét}} \simeq (\text{Stn}_{\mathbb{C}/t_0(X)}^{\text{der}})_{\text{ét}}$$

Proof. The universal property of étale morphisms of \mathcal{T}_{an} -structured topoi described in [12, Example 2.3.4] shows that both sides can be identified with the full subcategory of \mathcal{X} spanned by those objects $U \in \mathcal{X}$ such that $\mathcal{X}_{/U} \simeq \text{Sh}(S)$ for some topological space S which is the underlying space of a \mathbb{C} -analytic Stein space. \square

3.4. Essential surjectivity. We are now ready to complete the proof of Theorem 3.7. We will switch again to the context $(\mathrm{dAn}_{\mathbb{C}}^0, \tau, \mathbf{P}_{\text{ét}})$. We will need to discuss a variation of Proposition 3.15. Let X be a geometric stack for the context $(\mathrm{dAn}_{\mathbb{C}}^0, \tau, \mathbf{P}_{\text{ét}})$. We will denote by $X_{\text{ét}}$ its small étale site, that is the full subcategory of $(\mathrm{dAn}_{\mathbb{C}}^0)_{/X}$ spanned by étale morphisms (we refer to [20, Remark 2.17] for a definition). Furthermore, we have a fully faithful inclusion of sites

$$j: \mathrm{An}_{\mathbb{C}} \rightarrow \mathrm{dAn}_{\mathbb{C}}^0$$

which is cocontinuous in virtue of the same observation used to prove Corollary 3.14. In particular, it induces a restriction functor on the level of geometric stacks. We will denote such functor again by t_0 .

Proposition 3.16. *Let X be a geometric stack for the context $(\mathrm{dAn}_{\mathbb{C}}^0, \tau, \mathbf{P}_{\text{ét}})$. Then the functor $t_0: X_{\text{ét}} \rightarrow (t_0(X))_{\text{ét}}$ is an equivalence of sites.*

Proof. We prove this by induction on the geometric level of X . If X is (-1) -representable, this follows from Proposition 3.15. Suppose now that X is n -geometric and that the statement holds true for $(n-1)$ -geometric stacks. Choose an étale n -groupoid presentation U^\bullet for X . Recall that this means that U^\bullet is a groupoid object in the ∞ -category dSt , that each U^m is $(n-1)$ -geometric and that the map $U^0 \rightarrow X$ is $(n-1)$ -étale. Since t_0 commutes with products in virtue of Proposition 3.13 and takes effective epimorphisms to effective epimorphisms by [10, 7.2.1.14], we see that $V^\bullet := t_0(U^\bullet)$ is a groupoid presentation for $t_0(X)$.

Now, let $Y \rightarrow t_0(X)$ be an étale map. We see that $Y \times_{t_0(X)} V^\bullet \rightarrow V^\bullet$ is an étale map (i.e. it is a map of groupoids which is étale in each degree). By the inductive hypothesis, we obtain a map of simplicial objects $Z^\bullet \rightarrow U^\bullet$, which is such that

$$t_0(Z^\bullet) = Y \times_{t_0(X)} V^\bullet$$

Since $Y \times_{t_0(X)} V^\bullet$ was a groupoid, the same goes for Z^\bullet (here we use again the equivalence guaranteed by the inductive hypothesis). The geometric realization of Z^\bullet provides us with an étale map $Z \rightarrow X$. Since t_0 preserves effective epimorphisms, we conclude that $t_0(Z) = Y$. This construction is functorial in Y , and it provides the inverse to the functor t_0 . \square

If X is a n -geometric stack with respect to the context $(\mathrm{dAn}_{\mathbb{C}}^0, \tau, \mathbf{P}_{\text{ét}})$, its truncation is a $(n+1)$ -truncated, as it follows from the same argument given in [26, Lemma 2.1.1.2]. It follows that the mapping spaces in $X_{\text{ét}}$ are $(n+1)$ -truncated and therefore $X_{\text{ét}}$ itself is equivalent to an $(n+1)$ -category. It follows that the category of (non hypercomplete) ∞ -sheaves $\mathcal{X} := \mathrm{Sh}(X_{\text{ét}}, \tau)$ is an $(n+1)$ -localic topoi. Consider the composition

$$\mathcal{T}_{\text{an}} \times (X_{\text{ét}})^{\text{op}} \rightarrow \mathrm{dAn}_{\mathbb{C}}^0 \times (\mathrm{dAn}_{\mathbb{C}}^0)^{\text{op}} \xrightarrow{y} \mathcal{S}$$

where the last arrow is the functor classifying the Yoneda embedding (see [15, § 5.2.1]). This induces a well defined functor

$$\mathcal{O}_X : \mathcal{T}_{\text{an}} \rightarrow \text{PSh}(X_{\text{ét}})$$

and Corollary 3.4 shows that is hypercomplete.

Lemma 3.17. *Keeping the above notations, \mathcal{O}_X commutes with products, admissible pullbacks and takes τ -covers to effective epimorphisms. In other words, \mathcal{O}_X defines a \mathcal{T}_{an} -structure on \mathcal{X} .*

Proof. The functor $\mathcal{T}_{\text{an}} \rightarrow \text{dAn}_{\mathbb{C}}^0$ commutes with products and admissible pullbacks by Proposition 1.4. Moreover, it takes τ -covers to effective epimorphisms in virtue of Corollary 3.5. At this point, the conclusion is straightforward. \square

If $\{U_i \rightarrow X\}$ is an étale atlas of X , each U_i defines an object V_i in \mathcal{X} . Unraveling the definitions, we see that the \mathcal{T}_{an} -structured topos $(\mathcal{X}_{/V_i}, \mathcal{O}_X|_{V_i})$ is canonically isomorphic to $U_i \in \text{dAn}_{\mathbb{C}}^0$ itself. Therefore $X' := (\mathcal{X}, \mathcal{O}_X)$ is a derived \mathbb{C} -analytic space.

We are left to prove that $\phi(X') \simeq X$. We can proceed by induction on the geometric level n of X . If $n = -1$, the statement is clear. If $n \geq 0$, the first part of the proof of Theorem 3.7 shows that the Čech nerve of $\coprod U_i \rightarrow X$ is a groupoid presentation for $\phi(X')$. Since ϕ commutes with Čech nerves of étale maps and their realizations (in virtue of Corollary 3.5), we conclude that $\phi(X')$ is equivalent to X itself. The proof of Theorem 3.7 is now achieved.

4. COHERENT SHEAVES ON A DERIVED \mathbb{C} -ANALYTIC SPACE

In this section we introduce the notion of coherent sheaf on a derived \mathbb{C} -analytic space. Let us start by recalling that given an ∞ -topos \mathcal{X} it is possible to define the category of $\mathcal{D}(\text{Ab})$ -valued sheaves as in [12, §1.1]. This ∞ -category inherits a symmetric monoidal structure from the one of $\mathcal{D}(\text{Ab})$. If $\mathcal{O}_{\mathcal{X}}$ is \mathbb{E}_{∞} -ring on \mathcal{X} , we can therefore form the ∞ -category $\mathcal{O}_{\mathcal{X}}\text{-Mod}$ of $\mathcal{O}_{\mathcal{X}}$ -modules. We refer to [14, Proposition 2.1.3] for the main properties of this ∞ -category.

Definition 4.1. Let X be a derived \mathbb{C} -analytic space. The ∞ -category $\text{Coh}(X)$ is the full subcategory of $\mathcal{O}_X^{\text{alg}}\text{-Mod}$ spanned by those $\mathcal{O}_X^{\text{alg}}$ -modules \mathcal{F} such that the cohomology sheaves $\mathcal{H}^i(\mathcal{F})$ are locally on \mathcal{X} coherent sheaves of $\pi_0(\mathcal{O}_X^{\text{alg}})$ -modules. We let $\text{Coh}^+(X)$ be the full subcategory of $\text{Coh}(X)$ spanned by those coherent $\mathcal{O}_X^{\text{alg}}$ -modules \mathcal{F} such that $\mathcal{H}^i(\mathcal{F}) = 0$ for all $i \ll 0$ (in cohomological notation). We let $\text{Coh}^+(X)$ be the full subcategory of $\text{Coh}(X)$ spanned by those coherent $\mathcal{O}_X^{\text{alg}}$ -modules \mathcal{F} such that $\mathcal{H}^i(\mathcal{F}) = 0$ for all $|i| \gg 0$.

Clearly, both $\mathrm{Coh}^+(X)$ and $\mathrm{Coh}^b(X)$ are stable subcategories of $\mathcal{O}_X\text{-Mod}$. Moreover, the cohomology sheaves \mathcal{H}^i allow to define a t -structure on $\mathrm{Coh}(X)$, and one has

$$\mathrm{Coh}^\heartsuit(X) \simeq \mathrm{Coh}^\heartsuit(t_0(X))$$

as it follows combining Proposition 3.15 and [14, Remark 2.1.5].

Remark 4.2. It could seem that Definition 4.1 is to some extent arbitrary because it doesn't take at all into account the additional analytic structure of \mathcal{O}_X . This is not quite true, but a justification of this fact is beyond the scope of the present article. We will come back to this subject in [19].

In order to freely use the results proved in [20] for underived higher analytic stacks, we will need to compare the two categories of sheaves on them. We already moved a first step in this direction with Corollary 3.14. We will now complete the task as follows. Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived \mathbb{C} -analytic space. It follows from the very definition of étale morphism of derived \mathbb{C} -analytic space that there exists a fully faithful functor

$$(\mathrm{Stn}_{\mathbb{C}/X}^{\mathrm{der}})_{\mathrm{\acute{e}t}} \rightarrow \mathcal{X}$$

defined by sending an étale map $Y \rightarrow X$ to the object $U \in \mathcal{X}$ such that $\mathcal{Y} \simeq \mathcal{X}/_U$. Since X admits an atlas made of derived Stein spaces, we conclude that \mathcal{X} is equivalent to the ∞ -category of sheaves on $((\mathrm{Stn}_{\mathbb{C}/X}^{\mathrm{der}})_{\mathrm{\acute{e}t}}, \tau)$. Therefore we obtain:

Proposition 4.3. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a localic 0-truncated derived \mathbb{C} -analytic space. Then the ∞ -category of \mathcal{O}_X -modules (resp. of coherent \mathcal{O}_X -modules) on X is canonically equivalent to the one on $\phi(X)$ in the sense of [20, §5.1].*

5. THE GRAUERT THEOREM FOR DERIVED STACKS

In this short section we explain how the Grauert theorem proved in [20] for underived higher Artin analytic stacks induces an analogous theorem for derived \mathbb{C} -analytic spaces.

Definition 5.1. Let $f: X \rightarrow Y$ be a morphism of derived \mathbb{C} -analytic spaces and suppose that X and Y are n -localic for some n . We will say that f is *proper* if $t_0(f): t_0(X) \rightarrow t_0(Y)$ is proper as morphism of higher Deligne-Mumford analytic stacks (in the sense of [20, Definition 4.8]).

Remark 5.2. As the formulation of a reasonable definition of proper map between higher analytic stacks has perhaps been the newest concept introduced in [20], it is worth of recalling it. In this remark, we will limit ourselves to the \mathbb{C} -analytic case. The definition is given by induction on the geometric level of the map $f: X \rightarrow Y$ and it relies on the feebler notion of *weakly proper map*. In the \mathbb{C} -analytic setting,

one can say that a morphism of analytic stacks $f: X \rightarrow Y$ with Y representable is weakly proper if for every Stein open subset $W \subseteq Y$ and every atlas $\{U_i\}_{i \in I}$ of X there exists a finite subset $I' \subset I$ such that $\{W \times_Y U_i\}_{i \in I'}$ is an atlas for $W \times_Y X$. This definition is vaguely reminiscent of the topological notion of compact space. It is not quite the definition adopted in [20], but it is equivalent: see Definition 4.5 loc. cit. for the original one and Lemma 6.2 for the equivalence with the one we reported here.

Once this notion is established, we say that a morphism $f: X \rightarrow Y$ of higher \mathbb{C} -analytic stacks is proper if it is weakly proper and separated (i.e. the diagonal, whose geometric level is strictly less than the one of f , is proper). For example, the stack BG with G a \mathbb{C} -analytic Lie group is proper if and only if the group G was compact to begin with.

Remark 5.3. It follows from [20, Lemma 4.12] and Proposition 3.13 that proper morphisms are stable under base change. It can be further proved that they are stable under composition.

Lemma 5.4. *Let X be a derived \mathbb{C} -analytic space and let $i: t_0(X) \rightarrow X$ be the inclusion of its truncation. Then Ri_* takes $\mathrm{Coh}^+(t_0(X))$ to $\mathrm{Coh}^+(X)$ and it is of cohomological dimension 0.*

Proof. Since X is a derived Deligne-Mumford stack we can represent it as $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, where \mathcal{X} is the small étale topos of X . With this notation, we can identify $t_0(X)$ with $(\mathcal{X}, \pi_0(\mathcal{O}_{\mathcal{X}}))$. The functor Ri_* is therefore identified with the forgetful functor along $\mathcal{O}_{\mathcal{X}} \rightarrow \pi_0(\mathcal{O}_{\mathcal{X}})$. It follows immediately that Ri_* is of cohomological dimension 0 (in fact, it is t -exact). If $\mathcal{F} \in \mathrm{Coh}^+(t_0(\mathcal{X}))$, then each $\mathcal{H}^i(\mathcal{F})$ is a coherent $\pi_0(\mathcal{O}_{\mathcal{X}})$ -modules. Hence, it follows from the definitions that $\mathcal{F} \in \mathrm{Coh}^+(\mathcal{X})$. \square

With these definitions it is immediate to prove the following:

Proposition 5.5. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of derived Deligne-Mumford stacks. Suppose moreover that $t_0(\mathcal{Y})$ is locally noetherian. Then the derived push-forward*

$$Rf_*: \mathcal{O}_{\mathcal{X}}\text{-Mod} \rightarrow \mathcal{O}_{\mathcal{Y}}\text{-Mod}$$

takes the full subcategory $\mathrm{Coh}^+(X)$ to $\mathrm{Coh}^+(Y)$.

Proof. Let \mathcal{C} be the full subcategory of $\mathcal{O}_{\mathcal{X}}\text{-Mod}$ spanned by those $\mathcal{O}_{\mathcal{X}}$ -modules \mathcal{F} such that $Rf_*(\mathcal{F}) \in \mathrm{Coh}^+(Y)$. We make the following remarks:

- (1) \mathcal{C} is closed under loops and suspensions in $\mathcal{O}_{\mathcal{X}}\text{-Mod}$: this is obvious, since Rf_* is an exact functor of stable ∞ -categories;

- (2) \mathcal{C} is closed under extensions in $\mathcal{O}_X\text{-Mod}$: again, this follows from the fact that Rf_* takes fiber sequences to fiber sequences (being an exact functor of stable ∞ -categories);
- (3) \mathcal{C} contains $\text{Coh}^\heartsuit(X)$. Indeed, we have the following commutative square:

$$\begin{array}{ccc} t_0(\mathcal{X}) & \xrightarrow{f_0} & t_0(\mathcal{Y}) \\ \downarrow i & & \downarrow j \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

which induces a commutative square

$$\begin{array}{ccc} \pi_0(\mathcal{O}_X)\text{-Mod} & \xrightarrow{Rf_{0*}} & \pi_0(\mathcal{O}_Y)\text{-Mod} \\ \downarrow Ri_* & & \downarrow Rj_* \\ \mathcal{O}_X\text{-Mod} & \xrightarrow{Rf_*} & \mathcal{O}_Y\text{-Mod} \end{array}$$

If $\mathcal{F} \in \text{Coh}^\heartsuit(\mathcal{X})$, we can write $\mathcal{F} = Ri_*(\mathcal{F}')$ with $\mathcal{F}' \in \text{Coh}^\heartsuit(t_0(\mathcal{X}))$, as it follows from [14, Remark 2.1.5]. Therefore [20, Theorem 5.11] shows that $Rf_{0*}(\mathcal{F}') \in \text{Coh}^+(t_0(\mathcal{Y}))$, and Lemma 5.4 shows that $Rj_*(Rf_{0*}(\mathcal{F}')) \in \text{Coh}^+(\mathcal{Y})$.

To conclude the proof, we only need to observe that if $\tau_{\leq n}\mathcal{F} \in \text{Coh}^+(\mathcal{X}) \cap \mathcal{C}$ for every n , then $\mathcal{F} \in \mathcal{C}$. For a fiber sequence

$$\tau_{\leq n}\mathcal{F} \rightarrow \mathcal{F} \rightarrow \tau_{> n}\mathcal{F}$$

As Rf_* is an exact functor of stable ∞ -categories, we have a fiber sequence

$$Rf_*(\tau_{\leq n}\mathcal{F}) \rightarrow Rf_*\mathcal{F} \rightarrow Rf_*(\tau_{> n}\mathcal{F})$$

As Rf_* is left t -exact, we see that $\mathcal{H}^i(Rf_*(\tau_{> n}\mathcal{F}))$ vanishes whenever $i \leq n$. Therefore the long exact sequence of cohomology groups shows that

$$\mathcal{H}^i(Rf_*(\tau_{\leq n}\mathcal{F})) \rightarrow \mathcal{H}^i(Rf_*\mathcal{F})$$

is an equivalence for every $i \leq n$. Since $\tau_{\leq n}\mathcal{F} \in \text{Coh}^+(\mathcal{F})$, letting n vary, we obtain that the cohomology sheaves of $Rf_*\mathcal{F}$ are coherent. In other words, $\mathcal{F} \in \mathcal{C}$, and the proof is now complete. \square

6. THE ANALYTIFICATION FUNCTOR

As we explained in the introduction, this section contains the most important result of this article, from a technical point of view. Following [11, Remark 12.26],

we consider the morphism of pregeometries $\mathcal{T}_{\text{ét}} \rightarrow \mathcal{T}_{\text{an}}$ induced by the classical analytification functor of [9, Exposé XII]. This morphism induces a forgetful functor

$$(-)^{\text{alg}}: \mathcal{T}\text{op}(\mathcal{T}_{\text{an}}) \rightarrow \mathcal{T}\text{op}(\mathcal{T}_{\text{ét}})$$

defined informally by the rule $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{alg}})$. It follows from the general theory of [12, §2.1] that this functor admits a right adjoint, denoted by $\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}$. Moreover, [12, Proposition 2.3.18] shows that $\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}$ takes $\mathcal{T}_{\text{ét}}$ -schemes locally of finite presentation to \mathcal{T}_{an} -schemes locally of finite presentation. We can therefore invoke [11, Corollary 12.22] to conclude that $\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}$ takes derived Deligne-Mumford stacks locally of finite presentation to derived analytic spaces. In this section, we show that this analytification functor satisfies all the good properties one would expect. In particular, we will show that if (X, \mathcal{O}_X) is a classical scheme locally of finite type over \mathbb{C} , then $\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}(X, \mathcal{O}_X)$ can be canonically identified with the classical analytification of (X, \mathcal{O}_X) . Moreover, we will show that if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived Deligne-Mumford stack locally of finite presentation over \mathbb{C} , then the canonical map

$$\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})^{\text{alg}} \rightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

is flat. We will use these results to deduce derived versions of GAGA theorems in the next section.

6.1. Relative spectrum functor. Let us begin with a couple of general results concerning the relative spectrum functor associated to a morphism of geometries.

Proposition 6.1. *Let $\varphi: \mathcal{G}' \rightarrow \mathcal{G}$ be a morphism of geometries and suppose that both \mathcal{G}' and \mathcal{G} are compatible with n -truncations. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathcal{T}\text{op}(\mathcal{G}')$. Then the canonical morphism*

$$\text{Spec}_{\mathcal{G}'}^{\mathcal{G}}(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \rightarrow \text{Spec}_{\mathcal{G}'}^{\mathcal{G}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

Exhibits $\text{Spec}_{\mathcal{G}'}^{\mathcal{G}}(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ as n -truncation of $\text{Spec}_{\mathcal{G}'}^{\mathcal{G}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. In particular, it induces an equivalence on the underlying ∞ -topos.

Proof. Let $\mathcal{G}' \rightarrow \mathcal{G}'_{\leq n}$ and $\mathcal{G} \rightarrow \mathcal{G}_{\leq n}$ be n -stubs for \mathcal{G}' and \mathcal{G} respectively. The universal property defining n -stubs, implies the existence of a commutative square of morphism of geometries

$$\begin{array}{ccc} \mathcal{G}' & \xrightarrow{\varphi} & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{G}'_{\leq n} & \xrightarrow{\varphi_n} & \mathcal{G}_{\leq n} \end{array}$$

Therefore we have

$$\mathrm{Spec}_{\mathcal{G}}^{\mathcal{G}_{\leq n}} \circ \mathrm{Spec}_{\mathcal{G}'}^{\mathcal{G}} \simeq \mathrm{Spec}_{\mathcal{G}'_{\leq n}}^{\mathcal{G}_{\leq n}} \circ \mathrm{Spec}_{\mathcal{G}'}^{\mathcal{G}'_{\leq n}}$$

Combining this with Proposition 3.11, we obtain the desired result. \square

Furthermore we can prove the following result:

Proposition 6.2. *Let $\mathcal{G}, \mathcal{G}'$ be geometries compatible with n -truncations. Let $\varphi: \mathcal{G}' \rightarrow \mathcal{G}$ be a morphism of geometries and let $\mathcal{G}' \rightarrow \mathcal{G}'_{\leq n}$, $\mathcal{G} \rightarrow \mathcal{G}_{\leq n}$ be n -stubs for \mathcal{G}' and \mathcal{G} , respectively. The diagram*

$$\begin{array}{ccc} \mathrm{Sch}(\mathcal{G}'_{\leq n}) & \xrightarrow{\mathrm{Spec}_{\mathcal{G}'_{\leq n}}^{\mathcal{G}_{\leq n}}} & \mathrm{Sch}(\mathcal{G}_{\leq n}) \\ \downarrow & & \downarrow \\ \mathrm{Sch}(\mathcal{G}') & \xrightarrow{\mathrm{Spec}_{\mathcal{G}'}^{\mathcal{G}}} & \mathrm{Sch}(\mathcal{G}) \end{array}$$

commutes.

Proof. Since φ commutes with finite limits, the induced morphism

$$\tilde{\varphi}: \mathrm{Ind}((\mathcal{G}')^{\mathrm{op}}) \rightarrow \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$$

commutes with finite limits as well. In particular, we see that it takes k -truncated objects to k -truncated objects. The statement now follows from the following pair of observations:

- (1) if $A \in \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ is k -truncated, then, writing $\mathrm{Spec}^{\mathcal{G}}(A) = (\mathcal{X}_A, \mathcal{O}_A)$, \mathcal{O}_A is k -truncated;
- (2) if $A \in \mathrm{Ind}((\mathcal{G}')^{\mathrm{op}})$, then $\mathrm{Spec}_{\mathcal{G}'}^{\mathcal{G}}(\mathrm{Spec}^{\mathcal{G}}(A)) \simeq \mathrm{Spec}^{\mathcal{G}'}(\tilde{\varphi}(A))$.

As these statements follow directly from the definitions, the proof is complete. \square

Remark 6.3. We don't know whether it is possible to extend the above result to the category of \mathcal{G}' -structured topoi. In what follows, we won't need but the result we proved.

6.2. Comparison with the classical analytification. Consider the morphism of pregeometries $\mathcal{T}_{\mathrm{ét}} \rightarrow \mathcal{T}_{\mathrm{an}}$. The associated relative spectrum functor is by definition the analytification functor. Since it preserves the class of schemes locally of finite presentations, it defines a functor

$$\mathrm{Sch}^{\mathrm{f.p.}}(\mathcal{T}_{\mathrm{ét}}) \rightarrow \mathrm{Sch}^{\mathrm{f.p.}}(\mathcal{T}_{\mathrm{an}}) \subset \mathrm{dAn}_{\mathbb{C}}$$

The goal of this section is to show that it coincides with the classical analytification of Grothendieck.

Lemma 6.4. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived Deligne-Mumford stack and let $X^{\text{an}} = (\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}})$ be its analytification. Let $Y = (\mathcal{Y}, \mathcal{O}_Y) \rightarrow X$ be an étale morphism of Deligne-Mumford stacks. Then the analytification Y^{an} can be explicitly described as the pair $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$, where the ∞ -topos \mathcal{Z} is defined to be the pullback of*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{j_*} & \mathcal{X}^{\text{an}} \\ \downarrow & & \downarrow p_* \\ \mathcal{Y} & \xrightarrow{i_*} & \mathcal{X} \end{array}$$

while $\mathcal{O}_{\mathcal{Z}}$ is defined to be $j^{-1}\mathcal{O}_{\mathcal{X}^{\text{an}}}$. Suppose furthermore that $\mathcal{Y} \simeq \mathcal{X}/_U$ and that $U \rightarrow \mathbf{1}_X$ is an effective epimorphism. Then $\mathcal{Z} \simeq \mathcal{X}^{\text{an}}_{/p^{-1}U}$ and $p^{-1}U \rightarrow \mathbf{1}_{\mathcal{Z}}$ is an effective epimorphism.

Proof. The first part is just a reformulation of [12, Lemma 2.1.3]. The second part follows from the universal property of étale subtopoi (cf. [10, 6.3.5.8]) and from the fact that p^{-1} commutes with truncations and (therefore) with effective epimorphisms (see [10, 5.5.6.28]). \square

Proposition 6.5. *Let $X \in \mathcal{T}_{\text{ét}}$ be a smooth (derived) scheme. Then the analytification of X is 0-localic and 0-truncated, and it coincides with the classical analytification defined in [9].*

Proof. In virtue of Lemma 6.4, we only need to show that this results holds true for $X = \mathbb{A}_{\mathbb{C}}^n$, the algebraic n -dimensional affine space. If $(\mathcal{Y}, \mathcal{O}_Y)$ is any $\mathcal{T}_{\text{ét}}$ -structured topos, we have

$$\text{Hom}_{\text{L}\mathcal{T}_{\text{op}}(\mathcal{T}_{\text{ét}})^{\text{op}}}((\mathcal{Y}, \mathcal{O}_Y), \text{Spec}^{\mathcal{T}_{\text{ét}}}(\mathbb{A}_{\mathbb{C}}^n)) \simeq \mathcal{O}_Y(\mathbb{A}^1)^n$$

Suppose now that $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ is a derived \mathbb{C} -analytic space. Then, if we denote by $\mathcal{E}_{\mathbb{C}}^n$ the analytic n -dimensional affine space, we have

$$\text{Hom}_{\text{L}\mathcal{T}_{\text{op}}(\mathcal{T}_{\text{an}})^{\text{op}}}((\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}), \text{Spec}^{\mathcal{T}_{\text{an}}}(\mathcal{E}_{\mathbb{C}}^n)) \simeq \mathcal{O}_{\mathcal{Z}}(\mathcal{E}_{\mathbb{C}}^1)^n \simeq \mathcal{O}_{\mathcal{Z}}^{\text{alg}}(\mathbb{A}_{\mathbb{C}}^1)^n$$

We conclude that $\text{Spec}^{\mathcal{T}_{\text{an}}}(\mathcal{E}_{\mathbb{C}}^n)$ is the analytification of $\mathbb{A}_{\mathbb{C}}^n$. \square

Proposition 6.6. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a scheme locally of finite type over \mathbb{C} , seen as a derived scheme. Let $(\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}})$ be the analytification in the sense of [11]. Then $\mathcal{O}_{\mathcal{X}^{\text{an}}}$ is 0-truncated and moreover the canonical morphism*

$$(\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}}^{\text{alg}}) \rightarrow (\mathcal{X}, \mathcal{O}_X)$$

exhibits $(\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}}^{\text{alg}})$ as analytification of X in the sense of [9].

Proof. Choose a geometric envelope $\mathcal{T}_{\text{an}} \rightarrow \mathcal{G}_{\text{an}}$. The universal property defining \mathcal{G}_{an} shows that the morphism $\mathcal{T}_{\text{ét}} \rightarrow \mathcal{T}_{\text{an}}$ induces a (essentially) unique limit-preserving functor $\mathcal{G}_{\text{ét}} \rightarrow \mathcal{G}_{\text{an}}$. Since the analytification functor of [11] is defined to be the relative spectrum $\text{Spec}_{\mathcal{G}_{\text{ét}}}^{\mathcal{G}_{\text{an}}}$, the first statement follows directly from Proposition 6.2.

As for the second statement, we can assume X to be affine and therefore choose a closed immersion $X \rightarrow \mathbb{A}_{\mathbb{C}}^n$. Let $J \subset \mathbb{C}[X_1, \dots, X_n]$ be the ideal defining X . Choose generators $f_1, \dots, f_m \in J$ and consider the morphism $f: \mathbb{A}^n \rightarrow \mathbb{A}^m$ classified by these elements. We can therefore describe X as the truncation of the pullback

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{A}_{\mathbb{C}}^n \\ \downarrow & & \downarrow f \\ \text{Spec}(\mathbb{C}) & \xrightarrow{0} & \mathbb{A}_{\mathbb{C}}^m \end{array}$$

computed in $\text{dSch}_{\mathbb{C}}$. The analytification functor of [11] is a right adjoint. In particular, it commutes with limits. We can therefore identify $(\mathcal{X}^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$ with the pullback in $\mathcal{T}\text{op}(\mathcal{T}_{\text{an}})$

$$\begin{array}{ccc} (\mathcal{X}^{\text{an}}, \mathcal{O}_{X^{\text{an}}}) & \longrightarrow & \text{Spec}^{\mathcal{T}_{\text{an}}}(\mathcal{E}_{\mathbb{C}}^n) \\ \downarrow & & \downarrow \\ \text{Spec}^{\mathcal{T}_{\text{an}}}(\ast) & \longrightarrow & \text{Spec}^{\mathcal{T}_{\text{an}}}(\mathcal{E}_{\mathbb{C}}^m) \end{array}$$

Observe that the bottom horizontal morphism is a closed immersion. Therefore, this is a pullback in $\text{dAn}_{\mathbb{C}}$. In particular, we see that \mathcal{X}^{an} is a closed subtopos of $\mathcal{X}_{\mathcal{E}_{\mathbb{C}}^n}$, and it is therefore 0-localic, and inspection shows that it is the topos associated to the topological space X^{an} , the analytification of [9]. The result now follows from [11, Lemma 12.19]. \square

Corollary 6.7. *Let $f: X \rightarrow Y$ be a closed immersion of derived schemes locally of finite presentation over \mathbb{C} . Then $f^{\text{an}}: X^{\text{an}} \rightarrow Y^{\text{an}}$ is a closed immersion of derived \mathbb{C} -analytic spaces.*

Proof. In virtue of Proposition 6.1, it is enough to prove the statement when both X and Y are 0-truncated. In this case, the result is an immediate consequence of Proposition 6.6 and of [9, Exposé XII, Proposition 3.2]. \square

Let now $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a 0-truncated higher Deligne-Mumford stack, locally of finite presentation over $\text{Spec}(\mathbb{C})$. The functor $\psi: \text{Sch}(\mathcal{G}_{\text{ét}}^{\text{der}}(k)) \rightarrow \text{Sh}(\text{dAff}, \tau_{\text{qét}})$ of [12, Theorem 2.4.1] is fully faithful. It can be shown that if \mathcal{X} is n -localic, then $\psi(X)$ is a geometric Deligne-Mumford stack in the sense of [26] (see [18, Theorem 1.7] for a proof and a more precise comparison statement). On the other side,

$X^{\text{an}} = (\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}})$ defines via the functor $\phi: \mathbf{dAn}_{\mathbb{C}} \rightarrow \mathbf{Sh}(\mathbf{Stn}_{\mathbb{C}}^{\text{der}}, \tau)$ of Section 3.1 an analytic geometric stack. Since X was 0-truncated, Proposition 6.2 shows that X^{an} is 0-truncated and therefore Corollary 3.10 shows that we can identify $\phi(X^{\text{an}})$ with a higher Deligne-Mumford analytic stack in the sense of [20]. We can therefore compare $\phi(X^{\text{an}})$ with $\psi(X)^{\text{an}}$, where the latter is the analytification of $\psi(X)$ in the sense of [20, §6.1]. It is a rather easy task to show that the two notions coincide:

Proposition 6.8. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an n -localic 0-truncated higher Deligne-Mumford stack locally of finite presentation over $\text{Spec}(\mathbb{C})$. Keeping the above notations, there exists a natural isomorphism $\phi(X^{\text{an}}) \rightarrow \psi(X)^{\text{an}}$.*

Proof. To be clearer in this proof, let us explicitly write $\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}^{\text{an}}}(X)$ instead of X^{an} , and let us reserve the notation $(-)^{\text{an}}$ for the analytification functor of [20]. It follows from Lemma 6.4 that $\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}^{\text{an}}}(X)$ commutes with geometric realization of étale groupoids. The same is true for ϕ (see Corollary 3.5) and for ψ (see [12, 2.4.13]). Finally, $(-)^{\text{an}}$ is defined to be a left Kan extension, and therefore it has this property by construction. Therefore, it is sufficient to prove the statement when X is an affine scheme of finite presentation over $\text{Spec}(\mathbb{C})$, and in this case the statement follows directly from Proposition 6.6. \square

6.3. Flatness I. Let us begin with the following definition.

Definition 6.9. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be (connective) \mathbb{E}_{∞} -structured topoi (i.e. elements of $\mathbf{Top}(\mathcal{T}_{\text{disc}})$). We will say that a morphism $(f, \varphi): (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \rightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is *flat* if the $\varphi: f^{-1}\mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{O}_{\mathcal{X}}$ is a flat morphism of sheaves of \mathbb{E}_{∞} -rings.

Remark 6.10. Recall that a morphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ of (connective) \mathbb{E}_{∞} -sheaves on an ∞ -topos \mathcal{X} is said to be flat if the induced base change functor

$$- \otimes_{\mathcal{A}} \mathcal{B}: \mathcal{A}\text{-Mod} \rightarrow \mathcal{B}\text{-Mod}$$

is t -exact (with respect to the canonical t -structures of $\mathcal{A}\text{-Mod}$ and $\mathcal{B}\text{-Mod}$). If \mathcal{X} has enough points, this is equivalent to ask that for every geometric point $\eta^{-1}: \mathcal{X} \rightrightarrows \mathcal{S}: \eta_*$, the induced morphism $\eta^{-1}(\varphi): \eta^{-1}\mathcal{A} \rightarrow \eta^{-1}\mathcal{B}$ is a flat morphism of (connective) \mathbb{E}_{∞} -rings. In other words, the $\pi_0(\eta^{-1}(\varphi))$ is flat and one has equivalences

$$\pi_i(\eta^{-1}\mathcal{A}) \otimes_{\pi_0(\eta^{-1}\mathcal{A})} \pi_0(\eta^{-1}\mathcal{B}) \rightarrow \pi_i(\eta^{-1}\mathcal{B})$$

(that is, the morphism $\eta^{-1}(f)$ is strong).

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a derived Deligne-Mumford stack locally of finite presentation over \mathbb{C} and let us write

$$\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}^{\text{an}}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = (\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}})$$

Our goal is to show that the canonical morphism

$$(\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}}^{\text{alg}}) \rightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

is flat in the sense of Definition 6.9. This statement is local on the Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and therefore it will be sufficient to prove it for $\text{Spec}^{\mathcal{T}_{\text{ét}}}(A)$, where A is a connective \mathbb{E}_{∞} -ring of finite presentation over \mathbb{C} . The task will be greatly simplified if we could replace the $\mathcal{T}_{\text{ét}}$ -scheme $\text{Spec}^{\mathcal{T}_{\text{ét}}}(A)$ (whose underlying ∞ -topos is 1-localic) with the \mathcal{T}_{Zar} -scheme $\text{Spec}^{\mathcal{T}_{\text{Zar}}}(A)$ (whose underlying ∞ -topos is 0-localic). For this, we will need a digression on the relative spectrum associated to the morphism of pregeometries $\mathcal{T}_{\text{Zar}} \rightarrow \mathcal{T}_{\text{ét}}$.

Let \mathcal{X} be an ∞ -topos. Recall that the forgetful functor

$$\text{Str}_{\mathcal{T}_{\text{ét}}}^{\text{loc}}(\mathcal{X}) \rightarrow \text{Str}_{\mathcal{T}_{\text{Zar}}}^{\text{loc}}(\mathcal{X})$$

is fully faithful. If moreover we suppose that the hypercompletion \mathcal{X}^{\wedge} has enough points, then an hypercomplete object $\mathcal{O} \in \text{Str}_{\mathcal{T}_{\text{Zar}}}^{\text{loc}}(\mathcal{X}^{\wedge})$ belongs to the essential image of this functor if and only if all the stalks $\eta^{-1}\mathcal{O}$ are strictly henselian \mathbb{E}_{∞} -rings. Therefore, if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a $\mathcal{T}_{\text{ét}}$ -structure topos, we will denote by again by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ the associated \mathcal{T}_{Zar} -structured topos.

Proposition 6.11. *Let $A \in \text{CAlg}_{\mathbb{C}}$ be a connective \mathbb{C} -algebra. Then:*

- (1) $\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}(\text{Spec}^{\mathcal{T}_{\text{Zar}}}(A)) \simeq \text{Spec}^{\mathcal{T}_{\text{ét}}}(A)$.
- (2) $\text{Spec}_{\mathcal{T}_{\text{Zar}}}^{\mathcal{T}_{\text{an}}}(\text{Spec}^{\mathcal{T}_{\text{Zar}}}(A)) \simeq \text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}(\text{Spec}^{\mathcal{T}_{\text{ét}}}(A))$.
- (3) *the canonical map $q: \text{Spec}^{\mathcal{T}_{\text{ét}}}(A) \rightarrow \text{Spec}^{\mathcal{T}_{\text{Zar}}}(A)$ is flat in the sense of Definition 6.9.*

Proof. Statements (1) and (2) follow directly from the universal properties of the relative and absolute spectrum functors. We will prove statement (3). Let us denote by $\mathcal{O}_A^{\text{ét}}$ the $\mathcal{T}_{\text{ét}}$ -structure sheaf of $\text{Spec}^{\mathcal{T}_{\text{ét}}}(A)$ and by \mathcal{O}_A the \mathcal{T}_{Zar} -structure sheaf of $\text{Spec}^{\mathcal{T}_{\text{Zar}}}(A)$. We know that $\mathcal{O}_A^{\text{ét}}$ is an hypercomplete object of $\text{Sh}(A_{\text{ét}}, \tau_{\text{qét}})$ (see [13, Theorem 8.4.2.(3)]). Let $A \rightarrow B$ be an étale morphism. We can then factor it as $A \rightarrow A' \rightarrow B$, where $A \rightarrow A'$ is a Zariski open immersion, and one has $q^{-1}(\mathcal{O}_A)(B) = \mathcal{O}_A(A') = A'$. From this, we conclude that $q^{-1}\mathcal{O}_A$ is hypercomplete as well.

It follows from J. Lurie's version of Deligne's completeness theorem [13, Theorem 4.1] that the hypercompletion $\text{Sh}(A_{\text{ét}}, \tau_{\text{qét}})^{\wedge}$ has enough points, we can check the flatness of $\mathcal{O}_A^{\text{ét}} \rightarrow q^{-1}\mathcal{O}_A$ on stalks. If $\eta^{-1}: \text{Sh}(A_{\text{ét}}, \tau_{\text{qét}}) \rightleftarrows \mathcal{S}: \eta_*$ is a geometric point, [14, Proposition 1.1.6] shows that we can find a filtered diagram $\{A_{\alpha}\}$ of étale A -algebras such that for every $F \in \text{Sh}(A_{\text{acute}}, \tau_{\text{qét}})$ one has

$$\eta^{-1}(F) = \lim F(A_{\alpha})$$

Moreover, η^{-1} is canonically determined by a morphism $x: \operatorname{Spec}(k) \rightarrow \operatorname{Spec}(A)$ for some algebraically closed field k . It follows that

$$\eta^{-1}q^{-1}\mathcal{O}_A = A_x$$

where A_x denotes the Zariski stalk at the point x . On the other side, $A_x^{\text{s.h.}} = \eta^{-1}\mathcal{O}_A^{\text{ét}}$ is a strict henselianization of A_x . It follows that the canonical morphism $A_x \rightarrow A_x^{\text{s.h.}}$ is flat. This completes the proof. \square

Corollary 6.12. *Let A be a connective \mathbb{E}_∞ -algebra of finite presentation over \mathbb{C} . Let us write $\operatorname{Spec}_{\mathcal{T}_{\text{Zar}}}^{\mathcal{T}_{\text{an}}}(\operatorname{Spec}^{\mathcal{T}_{\text{Zar}}}(A)) = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Then the following are equivalent:*

- (1) *the morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{alg}}) \rightarrow \operatorname{Spec}^{\mathcal{T}_{\text{Zar}}}(A)$ is flat;*
- (2) *the morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{alg}}) \rightarrow \operatorname{Spec}^{\mathcal{T}_{\text{ét}}}(A)$ is flat.*

6.4. Analytification of germs. From this point on, we will focus on the Zariski analytification functor $\operatorname{Spec}_{\mathcal{T}_{\text{Zar}}}^{\mathcal{T}_{\text{an}}}$.

Let us recall that the analytification functor of [9, Exposé XII] is defined only for schemes locally of finite presentation over \mathbb{C} . One of the advantages of the relative spectrum functor introduced in [12] is that it can be applied also to germs of schemes, that is to \mathcal{T}_{Zar} -topoi of the form (\mathcal{S}, A) , where A is a local \mathbb{E}_∞ -algebra. We will turn our attention to the study of the \mathcal{T}_{an} -structured topos $\operatorname{Spec}_{\mathcal{T}_{\text{Zar}}}^{\mathcal{T}_{\text{an}}}(\mathcal{S}, A)$ in the special case where A arises as the ring of Zariski germs at one point of a derived (Zariski) scheme locally of finite presentation over \mathbb{C} .

Lemma 6.13. *Let $f: X \rightarrow Y$ be a morphism of topological spaces. Let $y \in Y$ be a closed point and suppose that $f^{-1}(y) = \{x\}$. Then the diagram*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\text{id}} & \mathcal{S} \\ \downarrow x_* & & \downarrow y_* \\ \operatorname{Sh}(X) & \xrightarrow{f_*} & \operatorname{Sh}(Y) \end{array}$$

is a pullback square in $\mathcal{T}\text{op}$.

Proof. The morphism y_* is a closed immersion of ∞ -topoi. Set $V := Y \setminus \{y\}$. Seeing V as a (-1) -truncated object in $\operatorname{Sh}(Y)$, we can identify $y_*: \mathcal{S} \rightarrow \operatorname{Sh}(Y)$ with the inclusion $\operatorname{Sh}(Y)/V \rightarrow \operatorname{Sh}(Y)$. The pullback along f_* can therefore be identified with $\operatorname{Sh}(X)/f^{-1}(V)$. Observe that $f^{-1}(V)$ can be identified with the (-1) -truncated object in $\operatorname{Sh}(Y)$ represented by the inverse image U of V along f . By hypothesis $X \setminus U = \{x\}$. It follows that $\operatorname{Sh}(X)/f^{-1}(V) \rightarrow \operatorname{Sh}(X)$ can be identified with $x_*: \mathcal{S} \rightarrow \operatorname{Sh}(X)$. The proof is now complete. \square

Let (X, \mathcal{O}_X) be a (Zariski) derived scheme locally of finite presentation over \mathbb{C} and let $(\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}})$ be its analytification in the sense of [11]. It follows from Proposition 6.6 that $\mathcal{X}^{\text{an}} = \text{Sh}(X^{\text{an}})$, where X^{an} is the underlying topological space of the analytification of (X, \mathcal{O}_X) in the sense of [9]. Let $x_*: \mathcal{S} \rightarrow \text{Sh}(X^{\text{an}})$ be a geometric point. The induced map

$$x^{-1}p^{-1}\mathcal{O}_X \rightarrow x^{-1}\mathcal{O}_{X^{\text{an}}}^{\text{alg}}$$

can be seen as a morphism

$$(\mathcal{S}, y^{-1}\mathcal{O}_{X^{\text{an}}}^{\text{alg}}) \rightarrow (\mathcal{S}, x^{-1}\mathcal{O}_X)$$

It follows from Lemma 6.13 and [12, Lemma 2.1.3] that we can identify, via the above morphism, $(\mathcal{S}, y^{-1}\mathcal{O}_{X^{\text{an}}})$ with the analytification of $(\mathcal{S}, x^{-1}\mathcal{O}_X)$.

We will now give a more explicit characterization of the \mathcal{T}_{an} -structure $\mathcal{O}_{X^{\text{an}}, y} := y^{-1}\mathcal{O}_{X^{\text{an}}}$ in term of $\mathcal{O}_{X, x} := x^{-1}\mathcal{O}_X$. Introduce the functor

$$\overline{\Psi}: \text{Str}_{\mathcal{T}_{\text{Zar}}}^{\text{loc}}(\mathcal{S})_{/\mathbb{C}} \rightarrow \text{Str}_{\mathcal{T}_{\text{an}}}^{\text{loc}}(\mathcal{S})_{/\mathcal{H}_0}$$

which is by definition the left adjoint to

$$\overline{\Phi}: \text{Str}_{\mathcal{T}_{\text{an}}}^{\text{loc}}(\mathcal{S})_{/\mathcal{H}_0} \rightarrow \text{Str}_{\mathcal{T}_{\text{Zar}}}^{\text{loc}}(\mathcal{S})_{/\mathbb{C}}$$

We have:

Lemma 6.14. *Keeping the above notations, $\mathcal{O}_{X^{\text{an}}, y} = \overline{\Psi}(\mathcal{O}_{X, x})$.*

Proof. We already argued that $(\mathcal{S}, \mathcal{O}_{X^{\text{an}}, y})$ is the analytification in the sense of [11] of $(\mathcal{S}, \mathcal{O}_{X, x})$. The universal property of the analytification shows therefore that for every $\mathcal{O} \in \text{Str}_{\mathcal{T}_{\text{an}}}^{\text{loc}}(\mathcal{S})_{/\mathcal{H}_0}$ we have

$$\begin{aligned} \text{Map}_{\text{Str}_{\mathcal{T}_{\text{an}}}^{\text{loc}}(\mathcal{S})_{/\mathcal{H}_0}}(\mathcal{O}_{X^{\text{an}}, y}, \mathcal{O}) &\simeq \text{Map}_{\mathcal{T}_{\text{Op}}(\mathcal{T}_{\text{an}})}((\mathcal{S}, \mathcal{O}), (\mathcal{S}, \mathcal{O}_{X^{\text{an}}, y})) \\ &\simeq \text{Map}_{\mathcal{T}_{\text{Op}}(\mathcal{T}_{\text{Zar}})}((\mathcal{S}, \mathcal{O}^{\text{alg}}), (\mathcal{S}, \mathcal{O}_{X, x})) \\ &\simeq \text{Map}_{\text{Str}_{\mathcal{T}_{\text{Zar}}}^{\text{loc}}(\mathcal{S})_{/\mathbb{C}}}(\mathcal{O}_{X, x}, \mathcal{O}^{\text{alg}}) \end{aligned}$$

Therefore, we conclude that $\mathcal{O}' = \overline{\Psi}(\mathcal{O})$. □

We formulate the following conjecture generalizing Lemma 6.14:

Conjecture 6.15. *Let $\mathcal{O} \in \text{Str}_{\mathcal{T}_{\text{Zar}}}^{\text{loc}}(\mathcal{S})_{/\mathbb{C}}$. Introduce the functor*

$$\overline{\Psi}: \text{Str}_{\mathcal{T}_{\text{Zar}}}^{\text{loc}}(\mathcal{S})_{/\mathbb{C}} \rightarrow \text{Str}_{\mathcal{T}_{\text{an}}}^{\text{loc}}(\mathcal{S})_{/\mathcal{H}_0}$$

left adjoint to the underlying algebra functor. Then the unit $\mathcal{O} \rightarrow \overline{\Psi}(\mathcal{O})^{\text{alg}}$ exhibits $(\mathcal{S}, \overline{\Psi}(\mathcal{O}))$ as analytification of $(\mathcal{S}, \mathcal{O})$.

6.5. Flatness II. Let us denote by $\mathrm{CAlg}_{\mathbb{C}}^{\mathrm{loc}}$ the ∞ -category of local \mathbb{C} -algebras (with residue field \mathbb{C}). We have a canonical identification $\mathrm{CAlg}_{\mathbb{C}}^{\mathrm{loc}} \simeq \mathrm{Str}_{\mathcal{T}_{\mathrm{Zar}}}^{\mathrm{loc}}(\mathcal{S})_{/\mathbb{C}}$. Recall also from Definition 2.5 that we denote by $\mathrm{AnRing}_{\mathbb{C}}^{\mathrm{loc}}$ the ∞ -category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}}^{\mathrm{loc}}(\mathcal{S})_{/\mathcal{H}_0}$. The adjunction considered at the end of the previous section can be rewritten as

$$\overline{\Psi}: \mathrm{CAlg}_{\mathbb{C}}^{\mathrm{loc}} \rightleftarrows \mathrm{AnRing}_{\mathbb{C}}^{\mathrm{loc}}: \overline{\Phi}$$

Accordingly to the notations of [11], we will write $(-)^{\mathrm{alg}}$ to denote the functor $\overline{\Phi}$. There is a forgetful functor

$$\mathrm{CAlg}_{\mathbb{C}}^{\mathrm{loc}} \rightarrow \mathcal{S}$$

which admits a left adjoint. We will denote it by $\mathbb{C}[-]$. Observe that $\pi_0 \mathbb{C}[\{*\}]$ can be identified with the ring of germs $\mathbb{C}[T]_{(0)}$.

The universal properties involved show that there is a natural equivalence $\mathcal{H}\{-\} \simeq \overline{\Psi} \circ \mathbb{C}[-]$, where $\mathcal{H}\{-\}$ is the ∞ -functor associated to the functor introduced in Section 2.5. To further simplify notations, we will write A^{an} instead of $\overline{\Psi}(A)^{\mathrm{alg}}$. If $K \in \mathcal{S}$ is a space, we will further write $\mathbb{C}\{K\}$ instead of $\mathbb{C}[K]^{\mathrm{an}}$.

Lemma 6.16. *Suppose that $f: A \rightarrow B$ is a morphism in the ∞ -category $\mathrm{AnRing}_{\mathbb{C}}^{\mathrm{loc}}$ which is surjective on π_0 . Then for every other morphism $g: A \rightarrow C$, the forgetful functor $(-)^{\mathrm{alg}}: \mathrm{AnRing}_{\mathbb{C}}^{\mathrm{loc}} \rightarrow \mathrm{CAlg}_{\mathbb{C}}^{\mathrm{loc}}$ preserves the homotopy pushout of this diagram.*

Proof. We can see A , B and C as $\mathcal{T}_{\mathrm{an}}$ -structures on \mathcal{S} . The maps $A \rightarrow B$ and $A \rightarrow C$ define morphisms $(\mathcal{S}, B) \rightarrow (\mathcal{S}, A)$ and $(\mathcal{S}, C) \rightarrow (\mathcal{S}, A)$ in $\mathrm{Top}(\mathcal{T}_{\mathrm{an}})$. The result follows now from combining [11, Proposition 10.3, Lemma 11.10]. \square

Lemma 6.17. *For every $n \in \mathbb{N}$, the maps $\mathbb{C}[\Delta^n] \rightarrow \mathbb{C}\{\Delta^n\}$ and $\mathbb{C}[\partial\Delta^n] \rightarrow \mathbb{C}\{\partial\Delta^n\}$ are flat.*

Proof. The morphism $\mathbb{C}[\Delta^0] \rightarrow \mathbb{C}[\Delta^n]$ is an acyclic cofibration, and the same goes for $\overline{\Psi}(\mathbb{C}[\Delta^0]) \rightarrow \overline{\Psi}(\mathbb{C}[\Delta^n])$. In particular, $\mathbb{C}\{\Delta^0\} \rightarrow \mathbb{C}\{\Delta^n\}$ is a weak equivalence. Now we have $\pi_0(\mathbb{C}[\Delta^0]) = (\mathbb{C}[\Delta^0])_0 = \mathbb{C}[T]$, while $\pi_0(\mathbb{C}\{\Delta^0\}) = (\mathbb{C}\{\Delta^0\})_0 = \mathbb{C}\{T\}$. Since $\mathbb{C}[T] \rightarrow \mathbb{C}\{T\}$ is flat, the first statement follows at once.

Let us turn to the case of $\partial\Delta^n$. When $n = 0, 1$, the result is trivial. If $n \geq 2$, we can present $\partial\Delta^n$ as the (homotopy) pushout of a diagram of the form

$$\begin{array}{ccc} \coprod \Delta^{n-2} \sqcup \Delta^{n-2} & \xrightarrow{f} & \coprod \Delta^{n-1} \\ \downarrow & & \downarrow \\ \coprod \Delta^{n-2} & \longrightarrow & \partial\Delta^n \end{array}$$

Both the functors $\mathbb{C}[-]$ and $\overline{\Psi}(\mathbb{C}[-])$ preserve this pushout. Moreover, the morphism

$$\coprod \Delta^{n-2} \sqcup \Delta^{n-2} \rightarrow \coprod \Delta^{n-2}$$

is a (degreewise) epimorphism, and the functors $\mathbb{C}[-]$ and $\overline{\Psi}$ commute with degree-wise epimorphisms (being left adjoints). It follows that the induced morphism

$$\widehat{\bigotimes} \overline{\Psi}(\mathbb{C}[\Delta^{n-2}]) \widehat{\otimes} \overline{\Psi}(\mathbb{C}[\Delta^{n-2}]) \rightarrow \widehat{\bigotimes} \overline{\Psi}(\mathbb{C}[\Delta^{n-2}])$$

is a degreewise epimorphism. It is moreover an epimorphism on π_0 , and therefore the image via $\mathbb{C}\{-\}$ of the above homotopy pushout is still a homotopy pushout. In other words, the diagram

$$\begin{array}{ccc} \mathbb{C}\{\coprod \Delta^{n-2} \sqcup \Delta^{n-2}\} & \longrightarrow & \mathbb{C}\{\coprod \Delta^{n-1}\} \\ \downarrow & & \downarrow \\ \mathbb{C}\{\coprod \Delta^{n-2}\} & \longrightarrow & \mathbb{C}\{\partial \Delta^n\} \end{array}$$

is both a homotopy pushout and a strict pushout. Observe moreover that

$$\begin{array}{ccc} \mathbb{C}[\coprod \Delta^{n-2} \sqcup \Delta^{n-2}] & \longrightarrow & \mathbb{C}[\coprod \Delta^{n-2}] \\ \downarrow & & \downarrow \\ \mathbb{C}\{\coprod \Delta^{n-2} \sqcup \Delta^{n-2}\} & \longrightarrow & \mathbb{C}\{\coprod \Delta^{n-2}\} \end{array}$$

is a strict pushout, that this diagram is homotopy equivalent to a (strict) pushout diagram of discrete objects and that the left vertical map is flat. Therefore it is homotopy equivalent to a homotopy pushout and therefore it is a homotopy pushout itself. As consequence, we see that the map $\mathbb{C}[\partial \Delta^n] \rightarrow \mathbb{C}\{\partial \Delta^n\}$ can be identified with the homotopy pushout of the map $\mathbb{C}[\coprod \Delta^{n-1}] \rightarrow \mathbb{C}\{\coprod \Delta^{n-1}\}$, which is flat. It follows that $\mathbb{C}[\partial \Delta^n] \rightarrow \mathbb{C}\{\partial \Delta^n\}$ is flat as well, thus completing the proof. \square

Lemma 6.18. *The diagram*

$$\begin{array}{ccc} \mathbb{C}[\partial \Delta^n] & \longrightarrow & \mathbb{C}[t] \\ \downarrow & & \downarrow \\ \mathbb{C}\{\partial \Delta^n\} & \longrightarrow & \mathbb{C}\{z\} \end{array}$$

is a pushout in the category of connective \mathbb{E}_∞ -algebras over \mathbb{C} .

Proof. Let $R = \mathbb{C}\{\partial \Delta^n\} \otimes_{\mathbb{C}[\Delta^n]} \mathbb{C}[t]$. We have a canonical map $g: R \rightarrow \mathbb{C}\{z\}$. Lemma 6.17 shows that both $\mathbb{C}[t] \rightarrow R$ and $\mathbb{C}\{z\}$ are flat. Therefore R is discrete,

and it will be sufficient to show that g induces an isomorphism on π_0 . The computations of Proposition 2.37 and Proposition 2.38 show that

$$\pi_0(R) = \mathrm{Tor}_0^{\pi_0(\mathbb{C}[\partial\Delta^n])}(\pi_0(\mathbb{C}\{\partial\Delta^n\}), \pi_0(\mathbb{C}[t])) \simeq \mathbb{C}\{z\}.$$

The proof is thus completed. \square

Theorem 6.19. *Let A be a local (connective) \mathbb{E}_∞ -algebra over \mathbb{C} . Then the morphism $A \rightarrow A^{\mathrm{an}}$ is flat.*

Proof. We can obtain A as colimit of a (possibly transfinite) diagram $\{A_\alpha\}_{\alpha < \lambda}$, depicted as

$$A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_n \rightarrow \cdots$$

where $A_0 = \mathbb{C}$ each map $A_\alpha \rightarrow A_{\alpha+1}$ is the pushout of

$$\begin{array}{ccc} \mathbb{C}[\partial\Delta^m] & \longrightarrow & \mathbb{C}[\Delta^0] \\ \downarrow & & \downarrow \\ A_\alpha & \longrightarrow & A_{\alpha+1} \end{array}$$

We will prove by transfinite induction that $A_\alpha \rightarrow A_\alpha^{\mathrm{an}}$ is flat. When $\alpha = 0$ there is nothing to prove. Since the functor $(-)^{\mathrm{alg}}$ commutes with sifted colimits (thus in particular with sequential ones) and since flat morphisms are stable under filtered colimits, we only need to prove that $A_{\alpha+1} \rightarrow A_{\alpha+1}^{\mathrm{an}}$ is flat given that $A_\alpha \rightarrow A_\alpha^{\mathrm{an}}$ is.

Applying $\overline{\Psi}$ to the above diagram, we obtain a pushout

$$\begin{array}{ccc} \mathcal{H}\{\partial\Delta^m\} & \longrightarrow & \mathcal{H}\{\Delta^0\} \\ \downarrow & & \downarrow \\ \overline{\Psi}(A_n) & \longrightarrow & \overline{\Psi}(A_{n+1}) \end{array}$$

Proposition 2.38 shows that $\mathcal{H}\{\partial\Delta^m\} \rightarrow \mathcal{H}\{\Delta^0\}$ is a surjection on π_0 and therefore Lemma 6.16 guarantees that $(-)^{\mathrm{alg}}$ preserves this pushout. In particular, we obtain a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{C}[\partial\Delta^m] & \longrightarrow & \mathbb{C}[\Delta^0] \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ \mathbb{C}\{\partial\Delta^m\} & \longrightarrow & \mathbb{C}\{\Delta^0\} & & \\ \downarrow & & \downarrow & & \downarrow \\ A_n^{\mathrm{an}} & \longrightarrow & A_{n+1}^{\mathrm{an}} & & \\ \uparrow & \swarrow & \uparrow & \swarrow & \uparrow \\ & & A_n & \longrightarrow & A_{n+1} \end{array}$$

Lemma 6.18 shows that the top square is a pushout. Therefore, in the rectangle

$$\begin{array}{ccccc} \mathbb{C}[\partial\Delta^m] & \longrightarrow & A_n & \longrightarrow & A_n^{\text{an}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}[\Delta^0] & \longrightarrow & A_{n+1} & \longrightarrow & A_{n+1}^{\text{an}} \end{array}$$

both the left square and the outer one are pushouts. Therefore the same goes for the one on the right. Since $A_n \rightarrow A_n^{\text{an}}$ was flat by hypothesis, we conclude that the same goes for $A_{n+1} \rightarrow A_{n+1}^{\text{an}}$. \square

Corollary 6.20. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a derived Deligne-Mumford stack locally of finite presentation over \mathbb{C} and write $\text{Spec}^{\mathcal{T}_{\text{ét}}^{\text{an}}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = (\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}})$. Then the canonical map $p: (\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}}^{\text{alg}}) \rightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is flat (in the sense of Definition 6.9).*

Proof. The question being local on $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, we can assume $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \text{Spec}^{\mathcal{T}_{\text{ét}}}(A)$ for a connective \mathbb{E}_{∞} -algebra A of finite presentation over \mathbb{C} . It follows from Corollary 6.12 that it is enough to prove that the canonical map

$$(\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}}^{\text{alg}}) \rightarrow \text{Spec}^{\mathcal{T}_{\text{Zar}}}(A)$$

is flat. In this case, Proposition 6.2 and Proposition 6.6 show that \mathcal{X}^{an} is a 0-localic ∞ -topos and Lemma 3.2 shows that \mathcal{X}^{an} is hypercomplete and has enough points. We are therefore reduced to show that for every geometric point $x_*: \mathcal{S} \rightarrow \mathcal{X}^{\text{an}}$, the induced map

$$x^{-1}p^{-1}\mathcal{O}_{\mathcal{X}} \rightarrow x^{-1}\mathcal{O}_{\mathcal{X}^{\text{an}}}^{\text{alg}}$$

is flat. Therefore, we can invoke Lemma 6.14 to reduce to the case of Theorem 6.19. This completes the proof. \square

6.6. Computing the analytification. A first important application of this flatness result is that it allows to give a more concrete description of the analytification of a derived Deligne-Mumford stack in terms of the analytification of its truncation.

Before giving the details of this, though, we will need to recall some terminology. If A is a simplicial ring and M is a connective A -module one can form the split square-zero extension $A \oplus M$. This is rather an easy task if we are working in the model category sCRing . However, if we need to generalize it to a less elementary ∞ -category, things become substantially more complicated. We refer to [15, §7.3.4] for a detailed account of this construction. To stress the non-triviality of this construction, we prefer to suppress the notation $A \oplus M$ in virtue of $\Omega_A^{\infty}(M)$, which is reminiscent of the way the construction goes. The framework developed in loc. cit. applies as well to sheaves of connective \mathbb{E}_{∞} -rings on any ∞ -topos and to their category of modules, and we will be using it precisely in this setting.

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a Deligne-Mumford stack locally of finite presentation over \mathbb{C} . Proposition 6.1 shows that for every $n \geq 0$ the natural morphism

$$\mathrm{Spec}_{\mathcal{T}_{\mathrm{ét}}}^{\mathcal{T}_{\mathrm{an}}}(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \rightarrow \mathrm{Spec}_{\mathcal{T}_{\mathrm{ét}}}^{\mathcal{T}_{\mathrm{an}}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

exhibits $\mathrm{Spec}_{\mathcal{T}_{\mathrm{ét}}}^{\mathcal{T}_{\mathrm{an}}}(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ as the n -truncation of $\mathrm{Spec}_{\mathcal{T}_{\mathrm{ét}}}^{\mathcal{T}_{\mathrm{an}}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. We can therefore write

$$\mathrm{Spec}_{\mathcal{T}_{\mathrm{ét}}}^{\mathcal{T}_{\mathrm{an}}}(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) = (\mathcal{X}^{\mathrm{an}}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}})$$

Let us further denote by $p_*: \mathcal{X}^{\mathrm{an}} \rightrightarrows \mathcal{X}: p^{-1}$ the induced geometric morphism.

We know that $\tau_{\leq n} \mathcal{O}_{\mathcal{X}} \rightarrow \tau_{\leq n-1} \mathcal{O}_{\mathcal{X}}$ is a square-zero extension. There exists therefore a pullback diagram

$$\begin{array}{ccc} \tau_{\leq n} \mathcal{O}_{\mathcal{X}} & \longrightarrow & \tau_{\leq n-1} \mathcal{O}_{\mathcal{X}} \\ \downarrow & & \downarrow d \\ \tau_{\leq n-1} \mathcal{O}_{\mathcal{X}} & \longrightarrow & \Omega^{\infty}(\pi_n(\mathcal{O}_{\mathcal{X}})[n+1]) \end{array}$$

Applying the functor p^{-1} we obtain the pullback diagram

$$\begin{array}{ccc} \tau_{\leq n} p^{-1} \mathcal{O}_{\mathcal{X}} & \longrightarrow & \tau_{\leq n-1} p^{-1} \mathcal{O}_{\mathcal{X}} \\ \downarrow & & \downarrow p^{-1}(d) \\ \tau_{\leq n-1} p^{-1} \mathcal{O}_{\mathcal{X}} & \longrightarrow & \Omega^{\infty}(\pi_n(p^{-1} \mathcal{O}_{\mathcal{X}})[n+1]) \end{array}$$

Since the morphism $p^{-1} \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}}$ is flat, we can invoke Lemma A.3 to conclude that

$$\begin{array}{ccc} \tau_{\leq n} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}} & \longrightarrow & \tau_{\leq n-1} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}} \\ \downarrow & & \downarrow p^{-1}(d) \otimes_{p^{-1} \mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}} \\ \tau_{\leq n-1} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}} & \longrightarrow & \Omega^{\infty}(\pi_n(\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}})[n+1]) \end{array}$$

is a pullback square. In other words, the analytification of the n -th Postnikov invariant of $\mathcal{O}_{\mathcal{X}}$ is the n -th Postnikov invariant of $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}}$.

Corollary 6.21. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a Deligne-Mumford stack locally of finite presentation over \mathbb{C} . Let $(\mathcal{X}^{\mathrm{an}}, \mathcal{O})$ be the analytification of $(\mathcal{X}, \Omega_{\tau_{\leq n-1} \mathcal{O}_{\mathcal{X}}}^{\infty}(\pi_n(\mathcal{O}_{\mathcal{X}}[n+1])))$. Then $\mathcal{O}^{\mathrm{alg}} = \Omega_{\tau_{\leq n-1} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}}}^{\infty}(\pi_n(\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}}[n+1]))$.*

Proof. Indeed, we know that $\mathcal{O}^{\mathrm{alg}}$ is n -truncated and

$$\tau_{\leq n-1} \mathcal{O} \simeq \tau_{\leq n-1} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}}^{\mathrm{alg}} \simeq \tau_{\leq n-1} \Omega_{\tau_{\leq n-1} \mathcal{O}_{\mathcal{X}}}^{\infty}(\pi_n(\mathcal{O}_{\mathcal{X}}[n+1]))$$

Therefore $\mathcal{O}^{\mathrm{alg}}$ is determined by its n -th Postnikov invariant. The above discussion allows to identify it with the analytification of the n -th Postnikov invariant of

$\Omega_{\tau_{\leq n-1}\mathcal{O}_X}^\infty(\pi_n(\mathcal{O}_X)[n+1])$, which is the null derivation. Therefore the n -th Postnikov invariant of \mathcal{O}^{alg} is the null derivation as well, and therefore we see that \mathcal{O}^{alg} can be identified with the split square-zero extension of $\tau_{\leq n-1}\mathcal{O}$ by $\pi_n(\mathcal{O})[n+1] \simeq \pi_n(\mathcal{O}_{X^{\text{an}}}^{\text{alg}})$. The conclusion follows. \square

Corollary 6.22. *Let (X, \mathcal{O}_X) be a Deligne-Mumford stack locally of finite presentation over \mathbb{C} . Then the analytification functor $\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}$ preserves the pushout:*

$$\begin{array}{ccc} (X, \Omega^\infty(\pi_n(\mathcal{O}_X)[n+1])) & \longrightarrow & (X, \tau_{\leq n-1}\mathcal{O}_X) \\ \downarrow & & \downarrow \\ (X, \tau_{\leq n-1}\mathcal{O}_X) & \longrightarrow & (X, \tau_{\leq n}\mathcal{O}_X) \end{array}$$

Proof. Let us write $(X^{\text{an}}, \mathcal{O}) = \text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}}(X, \Omega^\infty(\pi_n(\mathcal{O}_X)[n+1]))$. We only need to check that the commutative diagram

$$\begin{array}{ccc} \tau_{\leq n}\mathcal{O}_{X^{\text{an}}} & \longrightarrow & \tau_{\leq n-1}\mathcal{O}_{X^{\text{an}}} \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}\mathcal{O}_{X^{\text{an}}} & \longrightarrow & \mathcal{O} \end{array}$$

is a pullback. Since $(-)^{\text{alg}}$ preserves pullbacks and it is conservative, it is enough to check that

$$\begin{array}{ccc} \tau_{\leq n}\mathcal{O}_{X^{\text{an}}}^{\text{alg}} & \longrightarrow & \tau_{\leq n-1}\mathcal{O}_{X^{\text{an}}}^{\text{alg}} \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}\mathcal{O}_{X^{\text{an}}}^{\text{alg}} & \longrightarrow & \mathcal{O}^{\text{alg}} \end{array}$$

is a pullback. This follows from the previous corollary and the discussion before it. \square

Remark 6.23. One could ask whether a similar description holds *before* passing to the underlying $\mathcal{T}_{\text{ét}}$ -structured topos. The answer is affirmative, but the proof requires quite a bit of machinery that is beyond the scope of the present article. We will return on this point in [19].

7. GAGA FOR DERIVED DELIGNE-MUMFORD STACKS

This section is devoted to the two GAGA theorems. Let us first introduce the notion of proper morphism of derived Deligne-Mumford stacks. As for the analogous notion for derived \mathbb{C} -analytic spaces, we will reduce via the truncation to the definition given in [20]. Let us briefly recall the definitions given there:

Definition 7.1 ([20, Definition 4.7]). A morphism $f: X \rightarrow Y$ of algebraic Deligne-Mumford stacks is said to be *weakly proper* if there exists an atlas $\{Y_i\}_{i \in I}$ of Y such that for every $i \in I$, there exists a scheme P_i proper over Y_i and a proper surjective Y_i -morphism from P_i to $X \times_Y Y_i$.

Definition 7.2 ([20, Definition 4.8]). We define by induction on $n \geq 0$.

- (i) An n -representable morphism of algebraic Deligne-Mumford stacks is said to be *separated* if its diagonal being an $(n-1)$ -representable morphism is proper.
- (ii) An n -representable morphism of algebraic stacks is said to be *proper* if it is separated and weakly proper.

We now introduce the following definition:

Definition 7.3. Let $f: X \rightarrow Y$ be a morphism of derived Deligne-Mumford stacks. We will say that f is *separated* (resp. *proper*) if $t_0(f)$ is separated (resp. proper) in the sense introduced above.

Remark 7.4 (Algebraic proper direct image theorem). Using [20, Theorem 5.11] as basis of the induction, the same proof given in Proposition 5.5 yields a version of the proper direct image theorem statement for proper morphisms of higher algebraic Deligne-Mumford stacks whose truncation is locally noetherian. This is probably a folklore result (as it was the aforementioned theorem), but we couldn't locate it in the literature.

7.1. GAGA 1. Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of derived Deligne-Mumford stacks locally of finite presentations over \mathbb{C} . We have the following commutative diagram in $\mathcal{T}op(\mathcal{T}_{\acute{e}t})$:

$$\begin{array}{ccc} (X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}^{\text{alg}}) & \xrightarrow{h_X} & (X, \mathcal{O}_X) \\ \downarrow f^{\text{an}} & & \downarrow f \\ (Y^{\text{an}}, \mathcal{O}_{Y^{\text{an}}}^{\text{alg}}) & \xrightarrow{h_Y} & (Y, \mathcal{O}_Y) \end{array}$$

which in turn induces a commutative diagram of stable ∞ -categories

$$\begin{array}{ccc} \mathcal{O}_X\text{-Mod} & \xleftarrow{Rh_{X*}} & \mathcal{O}_{X^{\text{an}}}^{\text{alg}}\text{-Mod} \\ \downarrow Rf_* & & \downarrow Rf_*^{\text{an}} \\ \mathcal{O}_Y\text{-Mod} & \xleftarrow{Rh_{Y*}} & \mathcal{O}_{Y^{\text{an}}}^{\text{alg}}\text{-Mod} \end{array}$$

Given $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$, adjoint nonsense produces a canonical map

$$\varphi_{\mathcal{F}}: (Rf_*\mathcal{F})^{\text{an}} \rightarrow Rf_*^{\text{an}}(\mathcal{F}^{\text{an}})$$

where $\mathcal{F}^{\text{an}} := h_X^*(\mathcal{F})$ (and we don't write $Lh_X^*(\mathcal{F})$ thanks to the flatness result Corollary 6.20). Similarly, we wrote $(Rf_*\mathcal{F})^{\text{an}}$ to denote $h_Y^*(Rf_*\mathcal{F})$.

Theorem 7.5. *Let $f: X \rightarrow Y$ be a proper morphism of derived Deligne-Mumford stacks locally of finite presentation over \mathbb{C} . For every $\mathcal{F} \in \text{Coh}^+(\mathcal{X})$, the canonical map*

$$\varphi_{\mathcal{F}}: (Rf_*\mathcal{F})^{\text{an}} \rightarrow Rf_*^{\text{an}}(\mathcal{F}^{\text{an}})$$

is an equivalence.

Proof. Let \mathcal{C} be the full subcategory of $\mathcal{O}_X\text{-Mod}$ spanned by those \mathcal{F} for which $\varphi_{\mathcal{F}}$ is an equivalence. We observe that:

- (1) \mathcal{C} is stable under loops, suspensions and extensions: indeed, this follows immediately from the fact that all the functors Rf_* , Rf_*^{an} , h_X^* and h_Y^* are exact functors between stable ∞ -categories;
- (2) \mathcal{C} contains $\text{Coh}^{\heartsuit}(X) \simeq \text{Coh}^{\heartsuit}(t_0(X))$. Indeed, we have the commutative diagram

$$\begin{array}{ccc} t_0(X) & \xrightarrow{f_0} & t_0(Y) \\ \downarrow i & & \downarrow j \\ X & \xrightarrow{f} & Y \end{array}$$

If $\mathcal{F} \in \text{Coh}^{\heartsuit}(X)$, we can write $\mathcal{F} \simeq Ri_*(\mathcal{F}')$ with $\mathcal{F}' \in \text{Coh}^{\heartsuit}(t_0(X))$ and therefore $Rf_*(\mathcal{F}) \simeq Rj_*(Rf_{0*}\mathcal{F})$. The GAGA theorem of [20, Theorem 7.3] combined with Proposition 6.8 and Proposition 4.3 shows that the canonical map

$$(Rf_{0*}(\mathcal{F}'))^{\text{an}} \rightarrow Rf_{0*}^{\text{an}}((\mathcal{F}')^{\text{an}})$$

is an equivalence. We are therefore left to prove that the theorem holds for the inclusion $j: t_0(Y) \rightarrow Y$. This follows combining Corollary 6.20 and Proposition 6.2.

At this point, the dévissage lemma [20, Lemma 5.10] shows that $\text{Coh}^b(\mathcal{X}) \subset \mathcal{C}$. To complete the proof, we only need to show that whenever $\tau_{\leq n}\mathcal{F} \in \text{Coh}^+(X) \cap \mathcal{C}$ for every n , then $\mathcal{F} \in \mathcal{C}$.

Form a fiber sequence

$$\tau_{\leq n}\mathcal{F} \rightarrow \mathcal{F} \rightarrow \tau_{>n}\mathcal{F}$$

Since Rf_* , Rf_*^{an} and $(-)^{\text{an}}$ are exact functors between stable ∞ -categories, we obtain a morphism of fiber sequences

$$\begin{array}{ccccc} (Rf_*(\tau_{\leq n}\mathcal{F}))^{\text{an}} & \longrightarrow & (Rf_*\mathcal{F})^{\text{an}} & \longrightarrow & (Rf_*(\tau_{>n}\mathcal{F}))^{\text{an}} \\ \downarrow & & \downarrow & & \downarrow \\ Rf_*^{\text{an}}(\tau_{\leq n}\mathcal{F})^{\text{an}} & \longrightarrow & Rf_*^{\text{an}}\mathcal{F}^{\text{an}} & \longrightarrow & Rf_*^{\text{an}}(\tau_{>n}\mathcal{F})^{\text{an}} \end{array}$$

Corollary 6.20 shows that $(\tau_{>n}\mathcal{F})^{\text{an}} \simeq \tau_{>n}(\mathcal{F}^{\text{an}})$ and therefore both $H^i(Rf_*(\tau_{>n}\mathcal{F})^{\text{an}})$ and $H^i(Rf_*^{\text{an}}(\tau_{>n}\mathcal{F})^{\text{an}})$ vanish for $i \leq n$. Since $\tau_{\leq n}\mathcal{F} \in \text{Coh}^b(X)$ by hypothesis, we see the morphism $(Rf_*\mathcal{F})^{\text{an}} \rightarrow Rf_*^{\text{an}}\mathcal{F}^{\text{an}}$ induces an isomorphism on the cohomology groups H^i for every $i \leq n$. Letting n vary, we conclude that $\varphi_{\mathcal{F}}$ is an equivalence, thus completing the proof. \square

7.2. GAGA 2.

Lemma 7.6. *Let X be a derived Deligne-Mumford stack locally of finite presentation over \mathbb{C} . Let $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ and suppose it can be written as*

$$\mathcal{F} \simeq \lim_{\leftarrow} \tau_{\geq -n}\mathcal{F}$$

Then the analytification $(-)^{\text{an}}$ commutes with this limit.

Proof. It follows from Corollary 6.20 that $(\tau_{\geq n}\mathcal{F})^{\text{an}} \simeq \tau_{\geq n}\mathcal{F}^{\text{an}}$. Therefore we have

$$\mathcal{F}^{\text{an}} \simeq \lim_{\leftarrow} \tau_{\geq -n}\mathcal{F}^{\text{an}} \simeq \lim_{\leftarrow} (\tau_{\geq -n}\mathcal{F})^{\text{an}}$$

completing the proof. \square

Theorem 7.7. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived Deligne-Mumford stack proper over \mathbb{C} . The analytification functor*

$$(-)^{\text{an}}: \mathcal{O}_X\text{-Mod} \longrightarrow \mathcal{O}_{X^{\text{an}}}^{\text{alg}}\text{-Mod}$$

restricts to an equivalence

$$(-)^{\text{an}}: \text{Coh}(X) \longrightarrow \text{Coh}(X^{\text{an}})$$

Proof. Let us start by proving fully faithfulness. Since $\mathcal{O}_X\text{-Mod}$ and $\mathcal{O}_{X^{\text{an}}}^{\text{alg}}\text{-Mod}$ are stable and \mathbb{C} -linear, they are canonically enriched over $\mathcal{D}(\text{Ab})$, see [8, Examples 7.4.14, 7.4.15]. We will denote by $\text{Map}_{\mathcal{O}_X}^{\mathcal{D}(\text{Ab})}$ and $\text{Map}_{\mathcal{O}_{X^{\text{an}}}^{\text{alg}}}^{\mathcal{D}(\text{Ab})}$ the enriched mapping spaces of these two categories, respectively. For every $\mathcal{F}, \mathcal{G} \in \mathcal{O}_X\text{-Mod}$ there is a natural map

$$\psi_{\mathcal{F}, \mathcal{G}}: \text{Map}_{\mathcal{O}_X}^{\mathcal{D}(\text{Ab})}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Map}_{\mathcal{O}_{X^{\text{an}}}^{\text{alg}}}^{\mathcal{D}(\text{Ab})}(\mathcal{F}^{\text{an}}, \mathcal{G}^{\text{an}})$$

and we want to prove that $\psi_{\mathcal{F}, \mathcal{G}}$ is an equivalence. Observe that when $\mathcal{F}, \mathcal{G} \in \text{Coh}^{\heartsuit}(X) \simeq \text{Coh}^{\heartsuit}(t_0(X))$ the statement follows from the analogous result for

higher Deligne-Mumford stacks proved in [20, Proposition 7.2]. The same extension argument given in loc. cit. shows that $\psi_{\mathcal{F}, \mathcal{G}}$ is an equivalence whenever $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}^b(X)$.

Let us now turn to the general case. Since $\mathcal{D}(\mathrm{Ab})$ is both left and right t -complete, it is enough to prove that for every integer $n \in \mathbb{Z}$, $\pi_{-n}\psi_{\mathcal{F}, \mathcal{G}}$ is an isomorphism of abelian groups. Recalling that

$$\begin{aligned}\pi_{-n} \mathrm{Map}_{\mathcal{O}_X}^{\mathcal{D}(\mathrm{Ab})}(\mathcal{F}, \mathcal{G}) &= \pi_0 \mathrm{Map}_{\mathcal{O}_X}^{\mathcal{D}(\mathrm{Ab})}(\mathcal{F}, \mathcal{G}[n]) \\ \pi_{-n} \mathrm{Map}_{\mathcal{O}_{X^{\mathrm{an}}}^{\mathrm{alg}}}^{\mathcal{D}(\mathrm{Ab})}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}) &= \pi_0 \mathrm{Map}_{\mathcal{O}_{X^{\mathrm{an}}}^{\mathrm{alg}}}^{\mathcal{D}(\mathrm{Ab})}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}[n])\end{aligned}$$

we see that it is enough to treat the case $n = 0$. Observe that $\pi_0 \mathrm{Map}_{\mathcal{O}_X}^{\mathcal{D}(\mathrm{Ab})}(\mathcal{F}, \mathcal{G})$ can be identified with the global sections of the cohomology sheaf $\mathcal{H}^0(\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$. Since \mathcal{F} and \mathcal{G} are hypercomplete objects (see [14, Proposition 2.3.21]), we can write

$$\mathcal{F} \simeq \mathrm{colim}_n \tau_{\leq n} \mathcal{F}, \quad \mathcal{G} \simeq \lim_m \tau_{\geq m} \mathcal{G}$$

where the limits and colimits are computed in $\mathcal{O}_X\text{-Mod}$. Using the fact that $(-)^{\mathrm{an}}$ commutes with all colimits and invoking Lemma 7.6, we conclude that

$$\mathcal{F}^{\mathrm{an}} \simeq \mathrm{colim}_n (\tau_{\leq n} \mathcal{F})^{\mathrm{an}}, \quad \mathcal{G}^{\mathrm{an}} \simeq \lim_m (\tau_{\geq m} \mathcal{G})^{\mathrm{an}}$$

Moreover, Corollary 6.20 shows that $(\tau_{\leq n} \mathcal{F})^{\mathrm{an}} \simeq \tau_{\leq n} \mathcal{F}^{\mathrm{an}}$ and $(\tau_{\geq m} \mathcal{G})^{\mathrm{an}} \simeq \tau_{\geq m} \mathcal{G}^{\mathrm{an}}$. We are therefore reduced to the case where $\mathcal{F} \in \mathrm{Coh}^-(X)$ and $\mathcal{G} \in \mathrm{Coh}^+(X)$. Suppose more precisely that $\mathcal{H}^i(\mathcal{F}) = 0$ for $i \geq n_0$ and $cH^j(\mathcal{G}) = 0$ for $j \leq m_0$. Then the same bounds hold for $\mathcal{F}^{\mathrm{an}}$ and $\mathcal{G}^{\mathrm{an}}$, so that we obtain:

$$\begin{aligned}\pi_0 \mathrm{Map}_{\mathcal{O}_X}^{\mathcal{D}(\mathrm{Ab})}(\mathcal{F}, \mathcal{G}) &\simeq \pi_0 \mathrm{Map}_{\mathcal{O}_X}^{\mathcal{D}(\mathrm{Ab})}(\tau_{\geq m_0-1} \mathcal{F}, \tau_{\leq n_0+1} \mathcal{G}) \\ \pi_0 \mathrm{Map}_{\mathcal{O}_{X^{\mathrm{an}}}^{\mathrm{alg}}}^{\mathcal{D}(\mathrm{Ab})}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}) &\simeq \pi_0 \mathrm{Map}_{\mathcal{O}_X}^{\mathcal{D}(\mathrm{Ab})}(\tau_{\geq m_0-1} \mathcal{F}^{\mathrm{an}}, \tau_{\leq n_0+1} \mathcal{G}^{\mathrm{an}})\end{aligned}$$

Since both $\tau_{\geq m_0-1} \mathcal{F}$ and $\tau_{\leq n_0+1} \mathcal{G}$ belong to $\mathrm{Coh}^b(X)$, we already know that the canonical map

$$\mathrm{Map}_{\mathcal{O}_X}^{\mathcal{D}(\mathrm{Ab})}(\tau_{\geq m_0-1} \mathcal{F}, \tau_{\leq n_0+1} \mathcal{G}) \rightarrow \mathrm{Map}_{\mathcal{O}_X}^{\mathcal{D}(\mathrm{Ab})}(\tau_{\geq m_0-1} \mathcal{F}^{\mathrm{an}}, \tau_{\leq n_0+1} \mathcal{G}^{\mathrm{an}})$$

is an equivalence. In conclusion, $\psi_{\mathcal{F}, \mathcal{G}}$ is an equivalence for every $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$, completing the first part of the proof.

Let us now turn to the essential surjectivity part. Let \mathcal{C} be the full subcategory of $\mathrm{Coh}^b(X^{\mathrm{an}})$ spanned by the essential image of the analytification functor $(-)^{\mathrm{an}}$. Since $\mathrm{Coh}^\heartsuit(X^{\mathrm{an}}) \simeq \mathrm{Coh}^\heartsuit(t_0(X^{\mathrm{an}}))$, Proposition 6.8 and Proposition 4.3 can be combined together with the GAGA theorem of [20, Theorem 7.3] to conclude that \mathcal{C} contains $\mathrm{Coh}^\heartsuit(X^{\mathrm{an}})$. Observe that \mathcal{C} is clearly stable under loop and suspensions

in $\mathcal{O}_X\text{-Mod}$. We claim that \mathcal{C} is a thick subcategory of $\mathcal{O}_X\text{-Mod}$. Indeed, if we are given a fiber sequence

$$\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$$

with $\mathcal{F}', \mathcal{F}'' \in \mathcal{C}$, we can rotate it and express \mathcal{F} as the fiber of $\mathcal{F}'' \rightarrow \mathcal{F}'[1]$. Let $\mathcal{G}', \mathcal{G}'' \in \text{Coh}^b(X)$ be such that $(\mathcal{G}')^{\text{an}} \simeq \mathcal{F}'$ and $(\mathcal{G}'')^{\text{an}} \simeq \mathcal{F}''$. By the fully faithfulness we already proved, the map $\mathcal{F}'' \rightarrow \mathcal{F}'[1]$ is the analytification of a map $\mathcal{G}'' \rightarrow \mathcal{G}'[1]$. Let \mathcal{G} be the fiber of this map. Since $(-)^{\text{an}}$ is an exact functor between stable ∞ -categories, we see that $\mathcal{G}^{\text{an}} \simeq \mathcal{F}$, thus completing the proof of the claim. Therefore, the hypotheses of the dévissage lemma [20, Lemma 5.10] are satisfied and therefore we conclude that \mathcal{C} contains the whole $\text{Coh}^b(X^{\text{an}})$. If now $\mathcal{F} \in \text{Coh}^-(X^{\text{an}})$, we can write

$$\mathcal{F} \simeq \lim \tau_{\geq n} \mathcal{F}$$

Since the analytification functor $(-)^{\text{an}}$ is fully faithful and essentially surjective on $\text{Coh}^b(X^{\text{an}})$, we see that the diagram $\{\tau_{\geq n} \mathcal{F}\}$ is the analytification of a tower $\{\mathcal{G}_n\}$ in $\text{Coh}^b(X)$. Consider the morphism $\mathcal{G}_n \rightarrow \tau_{\geq n} \mathcal{G}_n$. Corollary 6.20 shows that it becomes an equivalence after passing to the analytification. Since $(-)^{\text{an}}$ is conservative, we conclude that $\mathcal{G}_n \in \text{Coh}^{\geq n}(X) \cap \text{Coh}^b(X)$. Therefore there are canonical maps $\tau_{\geq n} \mathcal{G}_{n-1} \rightarrow \mathcal{G}_n$, which become equivalences after applying $(-)^{\text{an}}$. These remarks show that $\mathcal{G} := \lim \mathcal{G}_n \in \text{Coh}^-(X)$. At this point, we can apply Lemma 7.6 to get

$$\mathcal{G}^{\text{an}} \simeq \lim \mathcal{G}_n^{\text{an}} \simeq \lim \tau_{\geq n} \mathcal{F} \simeq \mathcal{F}.$$

At last, this proves that $(-)^{\text{an}}$ is essentially surjective also on $\text{Coh}^-(X^{\text{an}})$, completing the proof. Since we already know that $(-)^{\text{an}}: \text{Coh}(X) \rightarrow \text{Coh}(X^{\text{an}})$ is fully faithful, we conclude now that $(-)^{\text{an}}: \text{Coh}^-(X) \rightarrow \text{Coh}^-(X^{\text{an}})$ is an equivalence.

Finally, let $\mathcal{F} \in \text{Coh}(X^{\text{an}})$. Repeating the same reasoning as before but using the truncations $\tau_{\leq n} \mathcal{F}$ and using the fact that $(-)^{\text{an}}$ commutes with limits, we finally conclude that there exists $\mathcal{G} \in \text{Coh}(X)$ such that $\mathcal{G}^{\text{an}} \simeq \mathcal{F}$. The proof is therefore complete. \square

8. EXTENSION TO ARTIN STACKS

In this section we outline how it is possible to extend all the results we obtained so far to the setting of derived Artin stacks.

We begin with a definition of derived Artin analytic stacks.

Definition 8.1. A morphism $f: X \rightarrow Y$ in $\text{Stn}_{\mathbb{C}}^{\text{der}}$ is said to be *smooth* if it is strong and its truncation $t_0(f)$ is smooth.

Let \mathbf{P}_{sm} be the collection of smooth morphisms in $\text{Stn}_{\mathbb{C}}^{\text{der}}$. Then the triple $(\text{Stn}_{\mathbb{C}}^{\text{der}}, \tau, \mathbf{P}_{\text{sm}})$ is a geometric context in the sense of [20]. We therefore give the following definition:

Definition 8.2. A *derived Artin analytic stack* is a geometric stack for the context $(\text{Stn}_{\mathbb{C}}^{\text{der}}, \tau, \mathbf{P}_{\text{sm}})$.

With this definition, the proof of the comparison results Corollary 3.10 and Proposition 4.3 remain unchanged. The analytification functor for derived Artin analytic stacks locally of finite presentation over \mathbb{C} is obtained as left Kan extension of the analytification functor

$$\text{Spec}_{\mathcal{T}_{\text{ét}}}^{\mathcal{T}_{\text{an}}} : \text{dAff}_{\mathbb{C}}^{\text{f.p.}} \rightarrow \text{Stn}_{\mathbb{C}}^{\text{der}}$$

exactly as it is done in [20, 24]. Therefore the comparison result Proposition 6.8 also extends to the setting of derived Artin stacks. We now observe that the proofs of our main theorems 5.5, 7.5 and 7.7 all rely on the analogous results of [20] by dévissage to the heart using the comparison results we cited above. Therefore the same technique applies to the setting of derived Artin stacks.

APPENDIX A. FLAT MORPHISMS

We collect in this section a couple of basic facts about flat morphisms of \mathbb{E}_{∞} -rings we couldn't find a reference for.

Lemma A.1. *Let $f: A \rightarrow B$ be a flat morphism of connective \mathbb{E}_{∞} -rings. Then $\tau_{\leq n}(A) \otimes_A B \simeq \tau_{\leq n}(B)$.*

Proof. It follows from [15, 7.2.2.13] that for every A -module M one has

$$\pi_i(M) \otimes_{\pi_0 A} \pi_0 B \simeq \pi_i(M \otimes_A B)$$

On the other side

$$\pi_i(M) \otimes_A B = \pi_i(M) \otimes_{\pi_0 A} \pi_0 A \otimes_A B$$

Since $A \rightarrow B$ is flat, we see that $\pi_0 A \otimes_A B$ is discrete and therefore there exists a map

$$\pi_0 B \rightarrow \pi_0 A \otimes_A B$$

and the spectral sequence of [15, 7.2.1.19] shows that this is indeed an isomorphism. In conclusion, we see that

$$\pi_i(M) \otimes_A B \simeq \pi_i(M \otimes_A B)$$

Now, since $A \rightarrow B$ is flat, we see that $\tau_{\leq n} A \otimes_A B$ is n -truncated, and therefore there exists a morphism

$$\tau_{\leq n} B \rightarrow \tau_{\leq n} A \otimes_A B$$

The above argument shows that this morphism induces isomorphisms on the π_i for every i and therefore it is an equivalence. \square

For the terminology used in the following lemma we refer back to the discussion at the beginning of Section 6.6.

Lemma A.2. *Let $f: A \rightarrow B$ be a morphism of \mathbb{E}_∞ -rings. Let M be an A -module. Then $\Omega_A^\infty(M) \otimes_A B \simeq \Omega_B^\infty(f^*(M))$.*

Proof. The functor $-\otimes_A B: \mathbf{CAlg}_{/A} \rightarrow \mathbf{CAlg}_{/B}$ commutes with finite limits (because finite limits are computed in the category $\mathbf{Sp}_{/A}$ and $\mathbf{Sp}_{/B}$ respectively and $-\otimes_A B$ is a functor between stable ∞ -categories). Therefore, the conclusion follows. \square

Lemma A.3. *Let $f: A \rightarrow B$ be a flat morphism of connective \mathbb{E}_∞ -rings. Let $d: \tau_{\leq n-1}A \rightarrow \Omega_A^\infty(\pi_n(A)[n+1])$ be the n -th Postnikov invariant of A . Then $d \otimes_{\tau_{\leq n-1}A} \tau_{\leq n-1}B: \tau_{\leq n-1}B \rightarrow \Omega_B^\infty(\pi_n(B)[n+1])$ is the n -th Postnikov invariant of B .*

Proof. The n -th Postnikov invariant of A is characterized by the fact that it makes the diagram

$$\begin{array}{ccc} \tau_{\leq n}A & \longrightarrow & \tau_{\leq n-1}A \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}A & \longrightarrow & \Omega_A^\infty(\pi_n(A)[n+1]) \end{array}$$

into a pullback diagram in the category of connective \mathbb{E}_∞ -rings. In particular, it is a pullback diagram in the category of $\tau_{\leq n}A$ -modules. Therefore the functor $-\otimes_{\tau_{\leq n}A} \tau_{\leq n}B$ preserves this pullback. Since $\tau_{\leq n-1}A \otimes_{\tau_{\leq n}A} \tau_{\leq n}B \simeq \tau_{\leq n-1}B$ and $\Omega_A^\infty(\pi_n(A)[n+1]) \otimes_{\tau_{\leq n}A} \tau_{\leq n}B \simeq \Omega_B^\infty(\pi_n(B)[n+1])$, we conclude that the diagram

$$\begin{array}{ccc} \tau_{\leq n}B & \longrightarrow & \tau_{\leq n-1}B \\ \downarrow & & \downarrow d \otimes_{\tau_{\leq n}A} \tau_{\leq n}B \\ \tau_{\leq n-1}B & \longrightarrow & \Omega_B^\infty(\pi_n(B)[n+1]) \end{array}$$

is a pullback square in the category of B -modules and, a posteriori, in the category of B -algebras. We conclude that $d \otimes_{\tau_{\leq n}A} \tau_{\leq n}B$ is the n -th Postnikov invariant of B . \square

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